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Boolean Delay Equations. II. Periodic and Aperiodic Solutions

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Boolean delay equations (BDEs) are *evolution equations* for a vector of discrete variables $\mathbf{x}(t)$. The value of each component $x_i(t)$, 0 or 1, depends on previous values of all components $x_j(t-t_{ij})$, $x_i(t)=f_i(x_1(t-t_{i1}),..., x_n(t-t_{in}))$. BDEs model the evolution of biological and physical systems with threshold behavior and nonlinear feedbacks. The delays model distinct interaction times between pairs of variables. In this paper, BDEs are studied by algebraic, analytic, and numerical methods. It is shown that solutions depend continuously on the initial data and on the delays. BDEs are classified into *conservative* and *dissipative*. All BDEs with rational delays only have *periodic solutions* only. But conservative BDEs with rationally unrelated delays have *aperiodic solutions* of increasing complexity. These solutions can be approximated arbitrarily well by periodic solutions of increasing period. *Self-similarity* and *intermittency* of aperiodic solutions is studied as a function of delay values, and certain number-theoretic questions related to *resonances* and diophantine approximation are raised. Period length is shown to be a lower semicontinuous function of the delays for a given BDE, and can be evaluated explicitly for linear equations. We prove that a BDE is *structurable stable* if and only if it has eventually periodic solutions of bounded period, and if the length of initial transients is bounded. It is shown that, for dissipative BDEs, asymptotic solution behavior is typically governed by a *reduced BDE*. Applications to *climate dynamics* and other problems are outlined.

KEY WORDS: Boolean functions; chaos; complexity; delay equations; difference equations; discrete evolution equations; dynamical systems; nonlinear feedbacks and threshold behavior.

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1. INTRODUCTION

In certain physical, as well as biological, systems interactions between state variables are highly nonlinear. For some of these systems, critical *thresholds* can be associated with the levels of interaction, as well as with the variables themselves. One may then describe the state of the system using a vector of Boolean variables, i.e., variables which take only the values 0 and 1. The interactions will be described by Boolean-valued functions of these Boolean variables. A variable with $k \geqslant 2$ discrete levels can be reexpressed by at most $k-1$ Boolean variables.

In the framework we envisage, the action of one Boolean state variable upon another is associated with a certain *delay*, representing the time it takes in the real-life system for the action to attain a critical threshold. Thus one is led to consider evolution equations for a vector of Boolean variables, with each variable depending upon the value of each of the other variables at some previous time; this past time depends upon the pair of variables acting and acted upon, respectively.

In molecular biology, the idea of thresholds and of a Boolean description was first formulated by Jacob and Monod.[1] Sugita[2] and Kauffman[3] formulated simple models with a single delay. The theory of cellular automata[4–6] and conservative logic[7,8] represent systematic generalizations of these ideas and models. Thomas[9,10] introduced multiple delays associated with the different time scales of action, and further expanded the theory.

In a typical example from genetic control theory, a set of interacting genes show a strong threshold behavior: a gene is "on" or "off." If a gene is "on," it produces a product which, in combination with the presence or absence of other gene products, can change the state of other genes. Furthermore, it takes a certain amount of time, i.e., a certain delay, before a gene product exists in sufficient quantity to have an effect on any other gene. The length of the delay depends upon the producing gene, as well as the affected gene.

Similar examples from population biology, chemical kinetics and electronic circuit theory are easy to construct. We give one more illustration from theoretical climate dynamics.[11,12] An elementary self-oscillatory model of quaternary glaciation cycles has two variables: global, annually averaged temperature T and ice volume V. T decreases as V increases, owing to the reduction in solar radiation absorbed by the system (the ice-albedo effect), while V decreases when T decreases, owing to the reduction of snow accumulation caused by a less active hydrological cycle (the precipitation–temperature effect). Again, the action of each effect mentioned above is associated with certain delays. These delays are distinct, being due either to the ocean's heat capacity or to the slow, viscoplastic flow of ice sheets, and are of the order of thousands of years each.

The mathematical framework of Boolean delay equations (BDEs), as discussed here, was introduced by Dee and Ghil[13] (hereafter BDE I). The initial-value problem of forward evolution was formulated, and proven to possess unique solutions for all times. Existence in the large required the evaluation of a certain bound on the complexity of solutions, which depends on the number of distinct delays. Continuous dependence on initial data and structural stability for BDEs were also introduced and discussed. Furthermore, a numerical example of BDE was given which exhibits aperiodic solutions of increasing complexity for certain delay values. This example motivated much of the work to be described here.

The *purpose of the present paper* is to continue the investigation of this class of equations started in BDE I. In Section 2, a group action on solutions is introduced. We show that BDEs which have only rational delays lead to solutions which are eventually periodic. BDEs are then classified in two ways: linear and nonlinear, and *conservative* and *dissipative*.

In Section 3, the *jump function* is introduced to count the number of jumps in a solution up to a certain time. An upper bound for the growth of this jump function was used in BDE I to prove existence in the large. A lower bound is obtained here to show that conservative BDEs with irrational delays have aperiodic solutions with growing complexity.

Section 4 contains an approximation theorem for solutions of BDEs. This theorem permits the numerical approximation of solutions and their jump functions using rational delays only. The self-similar behavior of solutions and their jump functions is exhibited and discussed. In Section 5, periodic solutions are examined more closely. Upper bounds for period length are computed explicitly for linear and nonlinear equations.

Dissipative BDEs are studied in Section 6. The connection between periodicity and structural stability is indicated. Asymptotic stability of solutions and nontrivial attractor sets are discussed.

Concluding remarks follow in Section 7. Possible applications and generalizations are outlined. Additional details on many aspects of BDEs can be found in Mullhaupt[14] (referred to subsequently as AM) and, for the sake of brevity, are not repeated here. In particular, two appendices in AM present: (a) the numerical methods used to compute solutions, their-jump functions and their periods and (b) continued fractions and their use in approximating irrational delays. Many proofs will be abridged or omitted entirely here, and the interested reader is referred to AM.

2. BOOLEAN DELAY EQUATIONS AND THEIR CLASSIFICATION

2.1. Boolean Delay Equations (BDEs)

A general system of delay equations for a vector $\mathbf{x} = (x_1, ..., x_n)$ is a set of equations

$$\begin{aligned} x_1(t) &= f_1(x_1(t - t_{11}), ..., x_n(t - t_{1n})) \\ &\vdots \qquad\qquad \vdots \\ x_n(t) &= f_n(x_1(t - t_{n1}), ..., x_n(t - t_{nn})) \end{aligned} \tag{2.1}$$

where $x_i: \mathbb{R} \to S$, $f_i: S^n \to S$, and S is some topological space. One normalizes the delays t_{ij} to be in the interval $(0, 1]$ and so that the largest one actually has unit value. Throughout this paper, we shall consider only autonomous, or closed, systems, with no explicit time dependence.

To study first the initial-value problem, one gives values in S^n for $\mathbf{x}(t)$ on $t \in [0, 1]$ and determines all possible continuations for $t > 1$ consistent with (2.1) and with the initial data. A unique function $\mathbf{x}(t)$ is determined that solves this problem, subject to certain conditions, and one wishes to study the qualitative dependence of this unique solution on the properties of the initial data, and on the delays. One is led to consider in general spaces of initial data compatible with the required properties of solutions (AM). Defining the space $S^n[0, 1]$ to be the space of S^n-valued compatible initial data, an endomorphism $\mathscr{T}_f$ acts on this space

$$\mathscr{T}_f: S^n[0, 1] \to S^n[0, 1] \tag{2.2a}$$

$$\mathscr{T}_f: \mathbf{x} \mid_{[n, n+1]} \mapsto \mathbf{x} \mid_{[n+1, n+2]} \tag{2.2b}$$

where $\mathbf{x}(t)$ is a solution to the delay equation (2.1).

The discrete-variable case $S = \mathbb{B} = \{0, 1\}$ is of interest in studying highly nonlinear systems which behave in a saturated manner, exhibiting finitely many recognizable states. The variables, as well as time, are also discretized in cellular automata theory,[3,4] kinetic logic,[9,10] and conservative logic.[8] As we shall see, a richer theory emerges by allowing time to remain continuous. The "synchronized" case of discrete time finds its place in a very natural way within this theory. The fact that synchronization can be of utmost importance in determining the dynamic behavior is one of the main points of this section.

In the discrete topology on *arbitrary* S, the only continuous functions $\mathbf{x}: [0, 1] \to S^n$ are constants, since $[0, 1]$ is connected. It is for this reason that piecewise continuous, i.e., piecewise constant, functions seem appropriate for our purposes. Given the discrete topology on any S, the only discontinuities are jumps between the discrete values of $\mathbf{x}(t)$. The number of jumps in the initial data is finite by the definition of piecewise continuity.

Jumps in the solution for $t > 1$ may only occur on a set of times given by translates of the initial jump times, through integral combinations of delays. This set was shown in BDE I to have no accumulation points. Thus the number of jumps in any finite interval is finite.

Changing the point of view from (2.1) to (2.2), $S^n[0, 1]$ is the *phase space* on which $\mathscr{T}_f$ acts. The number of jumps J_f in any iterate of the initial data under (2.2b) is finite. Thus J_f: $S^n[0, 1] \to \mathbb{N}$ is an integer-valued phase function.

In the Boolean case, $S = \mathbb{B} \equiv \{0, 1\}$ and $\mathscr{T}$ acts on $\mathbb{B}^n[0, 1]$. The proper topology is defined by the L^1 metric on $\mathbb{B}^1[0, 1]$, i.e.,

$$d(x, y) \equiv \int_0^1 \{x(t) \nabla y(t)\}\, dt = \int_0^1 |x(t) - y(t)|\, dt \tag{2.3}$$

where "∇" is the "exclusive or," $p \nabla q = (p \wedge \bar{q}) \vee (\bar{p} \wedge q)$, with $\bar{p} = not\, p$, and the second integral refers to x and y as real-valued functions.

In this topology $\mathscr{T}$ is continuous, but not Lipschitz continuous. In fact, the Lipschitz constant for $\mathscr{T}$ is just the jump function $\max\{J[\mathbf{x}(t)], J[\mathbf{y}(t)]\}$, which is not bounded on $\mathbb{B}^n[0, 1]$. Henceforth $\mathbb{B}^n[0, 1]$ equipped with the Boolean algebra and L^1-induced product topology is our phase space and will be denoted by X.

Theorem 2.1. $\mathscr{T}: X \to X$ is continuous for given delays.

Proof. See AM.

Theorem 2.2. $\mathscr{T}: X \times [0, 1]^{n^2} \to X \times [0, 1]^{n^2}$ is continuous, where $[0, 1]^{n^2}$ is the space of delays in the usual topology.

2.2. Rational Delays

We are ready to consider the asymptotic behavior of the dynamical system $\mathscr{T}^k$ as $k \to \infty$. The case of all delays being rational is the easiest.

Theorem 2.3. All the solutions of BDEs having only rational delays are eventually periodic.

Remark. The result holds for arbitrary initial data, including those with jumps at irrational points in $[0, 1]$ (see also the proof of Theorem 5.3).

Proof. Let q be the least common denominator (l.c.d.) of the delays, and define $\mathbf{x}_r$,

$$\mathbf{x}_r = x \mid_{[(r-1)/q, r/q]} \tag{2.4}$$

so that the set $\{\mathbf{x}_1,\ldots, \mathbf{x}_q\}$ are the initial data. Define recursively ϕ_k by

$$\begin{aligned}\mathbf{x}_{q+1} &= \phi_1(\mathbf{x}_1,\ldots, \mathbf{x}_q)\\ \mathbf{x}_{q+2} &= \phi_1(\mathbf{x}_2,\ldots, \mathbf{x}_q, \phi_1(\mathbf{x}_1,\ldots, \mathbf{q}_q))\\ &= \phi_2(\mathbf{x}_1,\ldots, \mathbf{x}_q)\end{aligned}$$

and so on,

$$\mathbf{x}_{q+k} = \phi_k(\mathbf{x}_1,\ldots, \mathbf{x}_q) \tag{2.5}$$

There are finitely many choices, say K, for ϕ_k, so in the sequence $\{\phi_k, k = 1, 2, 3,\ldots\}$ a "word" of length q must repeat. This word has length 1 in time and it determines uniquely the solution for all future time. The fact that the word repeats in its own continuation means that it must repeat again and again, and that the solution is periodic from the first repetition of this word on. ■

We use the word "periodic" to include solutions which are only periodic after an initial transient, and solutions which are eventually constant. *Aperiodic* then designates solutions which fail to be constant, periodic, or quasiperiodic on any time interval $[t, \infty)$.

The definitions (2.4) and (2.5) in the proof above suggest the need to consider q-tuples $\tau \in \mathbb{Z}_2^q$ and one-parameter families of such tuples, thus

$$\tau(s) = [x(s), x(s + 1/q),\ldots, x(s + (q-1)/q)] \tag{2.6}$$

The parameter is $s \in [0, 1/q)$ and tuple families will prove to be a useful concept in the sequel.

2.3. Conservative BDEs

In the case of (eventually) periodic solutions, what is the actual length of the transient? In certain systems of BDEs, transients are possible, and their presence or absence depends on the initial data. In other systems, when the delays are rational the periodicity begins immediately, for all initial data. We define such systems to form the class of *conservative* systems, and term all other systems of BDEs *dissipative*. The simplest nontrivial examples of a conservative and a dissipative BDE are

$$x(t) = \bar{x}(t-1)$$

and

$$x(t) = x(t-1) \wedge x(t-\theta)$$

respectively, with $0 < \theta < 1$.

Definition 2.1a. A system of Boolean delay equations is *conservative* for an open set $\Omega \subset (0, 1]^{n^2}$ of delays if for all rational delays in Ω and all initial data there are no transients.

Definition 2.1b. A *transient* is an initial state which is only visited once in the evolution of the system along a particular orbit in the phase space X.

The initial state in Definitions 2.1a, 2.1b above refers to a reduced phase space X_e of initial data for a given BDE with a given set of delays, $\{t_{ij}\}$. Let $e_i = 1 - \sup_k t_{ki}$.

Definition 2.1c. The *initial subspace* X_e of a BDE with delays $\{t_{ij}\}$ is defined by the restriction of elements $\mathbf{x}$ of X to intervals $[e_i, 1]$ for each component x_i.

It is common to call a Boolean function $\mathbf{f}$ a *connective*[15] and its arguments *channels*. Thus Definition 2.1c states that we only wish to look at the initial data of a channel x_i, or i for short, after its earliest appearance in determining the subsequent evolution of the system, for $t > 1$. Definition 2.1b states then that it suffices, for the purpose of defining conservative connectives, to consider transients from such restricted initial data.

Linear systems with rational delays. We address now the question of which initial data in X_e recur in the orbits they generate. In the case of *rational delays* with l.c.d. q, the space X_e can be represented as a one-parameter family of p-tuple spaces $\mathbb{Z}_2^p(s)$; cf. (2.6). Here p, with $q \leqslant p \leqslant nq$ represents the total number of $(1/q)$-long segments in the union over the n channels of the intervals $[e_i, 1]$ in Definition 2.1c, and $s \in [0, 1/q)$ as before. $\mathbb{Z}_2^p$ is a finite-dimensional vector space over $\mathbb{Z}_2$, and it is convenient sometimes to identify each element τ of $\mathbb{Z}_2^p$ with an element $\tau(\cdot)$ of $\mathbb{Z}_2^p(\cdot)$ which is constant on each $(1/q)$-long subinterval, i.e., $\tau'(s) = 0$, $\tau(s) \equiv \tau$.

The tuple families $\tau(\cdot)$ which generate nontransient states form a subset X_g of the family $\mathbb{Z}_2^p(\cdot)$. The tuple space family $\mathbb{Z}_2^r(\cdot)$, $r \leqslant p$, associated with X_g is invariant under the action of $\mathcal{T}_f$ on X, and it is in fact the maximal such invariant subspace of $\mathbb{Z}_2^p(\cdot)$. Therefore, we shall call X_g or $\mathbb{Z}_2^r(\cdot)$ also the *limit space.*

We start by studying this subset in the case in which $\mathscr{T}$ is a linear map on X. In speaking of linearity, we refer to addition (mod 2) in X, so that

$$x \oplus y = x + y \pmod 2 = x \bigtriangledown y$$

cf. (2.3), and will write $+$ for $\oplus$ when confusion is not imminent. Notice that $x \bigtriangledown y = 1$ if $x \neq y$ and 0 otherwise. Its negation is $x \bigtriangleup y = 1$ if $x = y$ and 0 otherwise, and

$$x \bigtriangleup y = 1 \oplus x \oplus y$$

Thus $x \bigtriangledown y$ corresponds to a *homogeneous* linear map from $\mathbb{Z}_2^2$ to $\mathbb{Z}_2$, and $x \bigtriangleup y$ to an inhomogeneous one.

$\mathscr{T}_f$ is linear iff (if and only if)

$$f_i = c_{i0} + \sum_1^n c_{ij} x_j(t - t_{ij}), \qquad 1 \leqslant i, j \leqslant n \tag{2.7}$$

where $c_{ij} \in \mathbb{Z}_2$, $0 \leqslant j \leqslant n$.

Lemma 2.1. Let Φ denote $\mathscr{T}^{1/q}$; cf. (2.5). As a linear map on $\mathbb{Z}_2^p(\cdot)$, Φ acts invertibly on $\mathbb{Z}_2^r(\cdot)$.

Proof. See AM.

Φ maps $\mathbb{Z}_2^p(\cdot)$ into itself, but not necessarily onto, and likewise for $\Phi(\mathbb{Z}_2^p(\cdot))$. Successive actions of Φ on $\mathbb{Z}_2^p(\cdot)$ yield a nested sequence of images

$$\mathbb{Z}_2^p \supset \Phi(\mathbb{Z}_2^p) \supset \cdots \supset \Phi^l(\mathbb{Z}_2^p) \supset \Phi^{l+1}(\mathbb{Z}_2^p) = \mathbb{Z}_2^r$$

where $\Phi^k(\mathbb{Z}_2^p) \neq \Phi^{k+1}(\mathbb{Z}_2^p)$ for $k \leqslant l$. The string of inclusions above defines the number l and shows that l/q is the maximum length of transients for the map $\mathscr{T} = \Phi^q$. The rank of Φ is thus r, and Φ has $\Delta = p - r$ zero eigenvalues. In this case, $\mathbf{f}$ will be conservative for a set of delays Ω iff $\Delta(\mathbf{f}; \{t_{ij}\}) = 0$ for *all* n^2-dimensional vectors of rational delays $\{t_{ij}\}$ in Ω.

The dimension p corresponding to X_e can be determined from the set $\{e_i\}$ of earliest appearances or, equivalently, from the set $\{\theta_i\}$ of longest delays for each x_i, $\theta_i = \sup_k t_{ki}$. Given the "cleared-fraction" delays $p_{ij} = q t_{ij}$, let the longest ones be $\bar{p}_i = q\theta_i$. The total number of bits in the tuple space $\mathbb{Z}_2^p$ is $p = \sum_1^n \bar{p}_i$.

To find r, the rank of Φ, it is easiest to compute the characteristic polynomial of Φ_r.[16,17]

The characteristic polynomial and its degree. First we want to reduce the system in order to eliminate the additive constants c_{i0}. This corresponds to subtracting a particular "inhomogeneous" solution of the linear system. It is convenient to choose this solution, $\mathbf{x}^0$ say, to lie in the limit set, $\mathbf{x}^0 \in \mathbb{Z}_2^r$. All solutions of the full system can be represented as the sum of $\mathbf{x}^0$ and of the solutions to the homogeneous system which we now study.

Define the *generating function* of this system's orbits componentwise by

$$G_i(z) = \sum_{-q}^{\infty} x_i(k)\, z^k \tag{2.8}$$

where we denote now $x_i(k/q)$ by $x_i(k)$, for brevity. By multiplying the linear system (2.1), (2.7) by z^k and summing over k, one can derive an equation for the generating function $\mathbf{G}(z) = (G_i(z))$,

$$A(z)\, \mathbf{G}(z) = \mathbf{D}(z; \tau_0) \tag{2.9a}$$

$$A_{ij} = \delta_{ij} + c_{ij} z^{p_{ij}} \tag{2.9b}$$

where $\tau_0 \in \mathbb{Z}_2^p$ is a p-tuple of initial data, and $D_i(z)$ are rational functions of z. Then

$$Q(z) = \det A(z) \tag{2.10}$$

is the characteristic polynomial of the matrix Φ for system (2.1), (2.7).

Solving explicitly for $\mathbf{G}$ we have

$$G_i(z) = \det A_i / Q(z) \tag{2.11a}$$

where A_i is the matrix $A(z)$ with the ith column replaced by D. In the scalar case

$$G(z) = D(z)/Q(z) \tag{2.11b}$$

In fact it is easy to show that the system is reducible to n scalar equations,

$$y(t) = y(t - r_1) + \cdots + y(t - r_k)$$

with integer delays $0 < r_1 < \cdots < r_k = q$. This can be written as a system, with $y(t) = y_k(t)$ and

$$y_i(t) = y_{i+1}(t-1), \qquad 1 \leqslant i \leqslant q-1 \tag{2.12a}$$

$$y_q(t) = \sum_{1}^{q} b_j\, y_j(t-1) \tag{2.12b}$$

Lemma 2.2. The characteristic polynomial of Φ for (2.12) is

$$Q(z) = 1 + \sum_1^q b_j z^j \tag{2.13a}$$

and the rank r of Φ is the degree of $Q(z)$,

$$r = \partial Q \tag{2.13b}$$

Comparing (2.11a) with (2.11b) shows that the evolution of each channel in the system, generated by $G_i(z)$, is given by the same characteristic polynomial (2.10), and by initial data $\det A_i(z; \tau_0)$ which are just linear combinations of the data for the entire system.

The rank of Φ for the general linear system (2.1), (2.7) is again the degree of $Q(z)$, $r = \partial Q$, as in (2.13b), while $p = \sum_i \bar{p}_i$. Hence we have the following result:

Theorem 2.4. A linear system of BDEs is conservative for an open neighborhood Ω of a fixed n^2 vector of rational, distinct delays $\{t_{ij}\}$ iff

$$\sum_i \bar{p}_i = \partial Q \tag{2.14}$$

for that vector of delays.

Proof. If the delays t_{ij} are all distinct, (2.14) means that the earliest appearance of each channel j has to occur in a component i of the connective (2.7) different from all the others. Hence, conservativity for linear systems depends on the ordering of the delays: an interchange of the largest delay between two channels can create transients or remove them. Such an interchange requires, however, a finite change in one or more delays. There exists therefore a small neighborhood of each delay for which no interchange occurs. ■

Remarks. (1) Here we encounter for the first time the importance of the relative lengths of delays for the qualitative behavior of a system of BDEs. More examples will occur in Section 6.

(2) An explicit treatment of transients in linear systems is given in AM.

Clearly, conservativity in the linear case, $p = r$, is equivalent to the invertibility of Φ, or the reversibility of the system of equations. This will prove to be the case, with a suitable change of the notion of invertibility, for nonlinear systems as well.

2.4. Conservativity, Reversibility and Invertibility

Definition 2.2. A system of BDEs is *reversible* if its time reversal also defines a system of BDEs.

A system is reversible if the value $x_i(e_i)$ of each channel x_i at its earliest appearance, e_i, as an argument in the evolution equations (2.1) can be deduced from the values of all channels at that time and at later times, $e_i \leqslant t \leqslant 1$. As in the linear case, there is a solvability part of this property which depends only on the connective, and another part which depends on the ordering of the delays. We shall simplify the discussion, as suggested by Theorem 2.4 above, in restricting it to the case of all delays being distinct from each other. Interesting phenomena which occur when delays become equal and "pass through each other" will be discussed in Section 6.

Excluding the trivial case of a channel which does not appear in any component of the connective $\mathbf{f}$, we denote by $\mathbf{x}(\mathbf{e})$ the n vector of earliest appearing components, $x_i(\mathbf{e}) = x_i(e_i)$.

Lemma 2.3. In order for a BDE system to be reversible, it is necessary that a given component f_i of the connective depend on at most one component of $\mathbf{x}(\mathbf{e})$.

Proof. See AM.

If the condition of the lemma holds, one can write the BDE (2.1) at $t = 1$ as

$$x_{\sigma(j)}(1) = f_{\sigma(j)}(x_j(1-\theta_j), x_k(1-t_{\sigma(j),k})), \qquad j = 1,..., n \tag{2.15}$$

where $k \neq j$ and $\sigma(j)$ designates the unique component of $\mathbf{f}$ in which the earliest appearance of channel j occurs.

To write the reversed system, we realign first each equation in (2.15) at $t = 0$,

$$x_{\sigma(j)}(1-e_j) = f_{\sigma(j)}(x_j(0), x_k(1-e_j-t_{\sigma(j),k})) \tag{2.16}$$

If the connective $\mathbf{f}$ satisfies the solvability condition to be determined momentarily, one can write, componentwise,

$$x_j(0) = (f^{-1})_j(x_{\sigma(j)}(1-e_j), x_k(1-t'_{jk})) \tag{2.17}$$

Here $(f^{-1})_j$ is the jth component of the inverse $\mathbf{f}^{-1}$ of $\mathbf{f}$, and $1 - t'_{jk}$ are the appropriate "backward delays" or "anticipations."

Lemma 2.4. Given that only one $x_j(e_j)$ appears in each f_i, $j = \pi(i)$, $\pi = \sigma^{-1}$ it is necessary for the system to be reversible that x_j appear *linearly* in f_i.

Proof. Fix all other delayed values $x_k(t-t_{ik})$, $k \neq j$, and consider a change in $x_j(e_j)$. Reversibility implies that $x_i(1)$ has to change also as a result. Let

$$g_i(t) = x_i(t) + x_j(t-\theta_j) \tag{2.18a}$$

Then

$$g_i(1) = x_i(1) + x_j(e_j) \tag{2.18b}$$

and

$$g_i(t) = f_i + x_j(t-t_{ij}) \tag{2.19}$$

The point is that x_j in f_i has to cancel by addition. ■

Definition 2.3. A system of BDEs is *invertible* if each component of the connective, f_i, contains exactly one component $x_j(e_j)$, $j=\pi(i)$, of the vector of earliest appearances of the channels, and if x_j appears linearly in f_i.

Theorem 2.5. Definitions 2.1, 2.2 and 2.3 are all three equivalent.

Proof. See AM.

Consider next a first-order system of BDEs, not necessarily linear,

$$x_i(t) = \sum_j c_{ij} x_j(t-t_{ij}) + g_i(x_{j'}(t-t_{ij'})), \qquad 1 \leqslant i \leqslant n, \tag{2.20}$$

where g_i contains only channels j' for which $c_{ij'}=0$. The *generalized characteristic polynomial* (*GCP*) of this system, for l.c.d. $\{t_{ij}: c_{ij} \neq 0\} = q$, is the characteristic polynomial $Q(z)$ given by (2.10) of the linear system obtained by setting $g_i \equiv 0$, $i=1,..., n$. For systems with rational delays in their linear part, we can give the following provisional definition of partial linearity.

Definition 2.4. System (2.20) is *partially linear* if $\partial Q \geqslant 2$.

Remark. The reason for not including in the definition the trivial case $\partial Q = 1$ will become clear in the next section, where the definition will be extended to irrational delays as well.

Theorem 2.6. A partially linear system of BDEs is conservative in a neighborhood Ω of a vector of rational delays $D = \{t_{ij}\}$ iff (2.14) holds for the degree of its GCP, defined for that D.

Proof. See AM.

The behavior of linear and partially linear systems is studied in the next section.

3. CONSERVATIVE BDEs AND APERIODIC SOLUTIONS

3.1. The Delay Lattice

Let Λ be the closure of the n^2 vector of delays $D = \{t_{ij}\}$ under the addition of real numbers, i.e.,

$$\Lambda = \left\{ \sum_{ij} k_{ij} t_{ij},\ t_{ij} \in D,\ k_{ij} \in \mathbb{N} \right\} \tag{3.1}$$

The dimension δ of Λ is the number of elements in a basis over the non-negative integers. The *delay lattice* Γ is the lattice over the integers generated by the basis obtained in this way.[18]

The delay lattice is useful in calculating the jump function, J_f: $\mathbb{B}^n[0, 1] \to \mathbb{N}$, introduced in Section 2.1. Here we shall consider specifically the number of jumps $J(k)$ in a unit interval $[k, k+1)$ for the solution corresponding to given initial data, $J(k; \mathbf{x} \mid_{[0,1)})$.

Assume for simplicity that there is a single jump at t_0 in the initial data. Then all of the jumps in the solution will occur at translates of t_0 by integral numbers of delays; in other words, the jumps will be on a subset of $t_0 + \Gamma$. Another way of saying this is that the domain of influence of a jump at t_0 is the translate of Γ by t_0.

The number of translates, i.e., of elements of Γ, less than a given time T is $O(T^\delta)$ and the growth of the number of translates per unit interval is $O(T^{\delta - 1})$. The fact that no connective can actually realize all possible jumps suggests that this is not a sharp estimate. To improve upon it requires, however, a hypothesis on the propagation of jumps by the connective.

3.2. Aperiodic Solutions of Conservative BDEs

The classification of the previous section provides us with a set of hypotheses which will facilitate the computation of $J(k)$, to within sharp bounds. The case of linear conservative systems is crucial in this computation.

In Section 2.4 we found that *nonlinear systems with rational delays* have a generalized characteristic polynomial (GCP) which characterizes a conservative linear system whose jumps are a subset of the jumps of the full, nonlinear system. In the case of partially linear systems, the characteristic polynomial of the linear part, i.e., the GCP, is nonconstant. We know that, for a conservative system, there is at least one way to assign the earliest appearance $x_j(e_j)$ of each channel j to each component f_i of the connective, $i = \sigma(j)$, and that x_j appears linearly in f_i. In fact, to each distinct assignment of earliest appearances, $\sigma = \pi^{-1}$, there corresponds a different nonconstant term, or monomial, in the GCP.

The number of nonconstant terms is the number of arguments in the connective of the equivalent scalar equation. As each argument appears linearly in it, one can assign the largest delay to either one of them, without affecting the equation's, and hence the system's, reversibility. The largest delay in this equation is associated with the monomial of maximal degree, r, which for a conservative system with rational delays equals p.

In this section we are interested mostly in *irrational delays*. It is necessary therefore to give a more general definition of the GCP. Let the linear part of system (2.20) be written

$$x_i(t) + \sum_j c_{ij} x_j(t - t_{ij}) = 0 \tag{3.2}$$

We associate with it a matrix

$$A(\lambda) = (\delta_{ij} + c_{ij} \lambda^{t_{ij}}) \tag{3.3a}$$

Definition 3.1. The GCP of system (2.20) is

$$Q(\lambda) = \det A(\lambda) \tag{3.3b}$$

Remark. If all the delays are rational, Eq. (3.3) reduces to the previous definition with $\lambda = z^q$ and $q = \text{l.c.d.}\{t_{ij}\}$.

The formal expression (3.3) serves to identify an equivalence class of systems of BDEs which are reversible iff $Q(\lambda)$ has a constant term. In this section and Section 4 only, the term GCP is used to designate (3.3b).

Our study of asymptotic properties of solutions of BDEs allows us to neglect transient behavior altogether, hence all of the information provided by characteristic polynomials can be found in the study of higher-order scalar linear equations. The delay lattice dimension δ for such equations is the number of rationally independent delays. We show that for $\delta > 1$ it is possible to give a lower bound for the jump function $J(k)$ which increases in time. Hence, the solutions to these scalar equations cannot be periodic or quasiperiodic.

Theorem 3.1. Consider the linear scalar equation

$$x(t) = x(t-1) \nabla x(t-\theta_2) \nabla \cdots \nabla x(t-\theta_\delta) \tag{3.4}$$

where $0 < \theta_\delta < \cdots < \theta_2 < \theta_1 = 1$ are rationally independent and $\delta \geqslant 2$. All solutions, except $x(t) \equiv 0$, are aperiodic and have complexity which increases with time.

Proof. The argument will be carried through for the case $\delta = 2$. We consider the scalar, second-order BDE

$$x(t) = x(t-1) \nabla x(t-\theta) \tag{3.5}$$

with θ irrational and a single jump in the initial data at $1-\theta < t_0 < 1$.

The delay lattice of (3.5) is shown in Fig. 1. The 1 axis is shown as the y abscissa, the θ axis as the z ordinate. Solid circles indicate where jumps actually occur in the solution, and constitute its *jump set*.

The solid line in the figure can be written as $y + z\theta = t$; it has slope $-\theta^{-1}$ and represents an *isochron*, i.e., the set of points in the diagram which corresponds to the time t. In the lattice Γ itself only one point can lie on an isochron. This is the main advantage of this reduced representation of the set Λ in (3.1). The calculation of the jump function $J(k)$ is equivalent to counting the number of solid circles which lie between the two isochrons $t = k$ and $t = k + 1$.

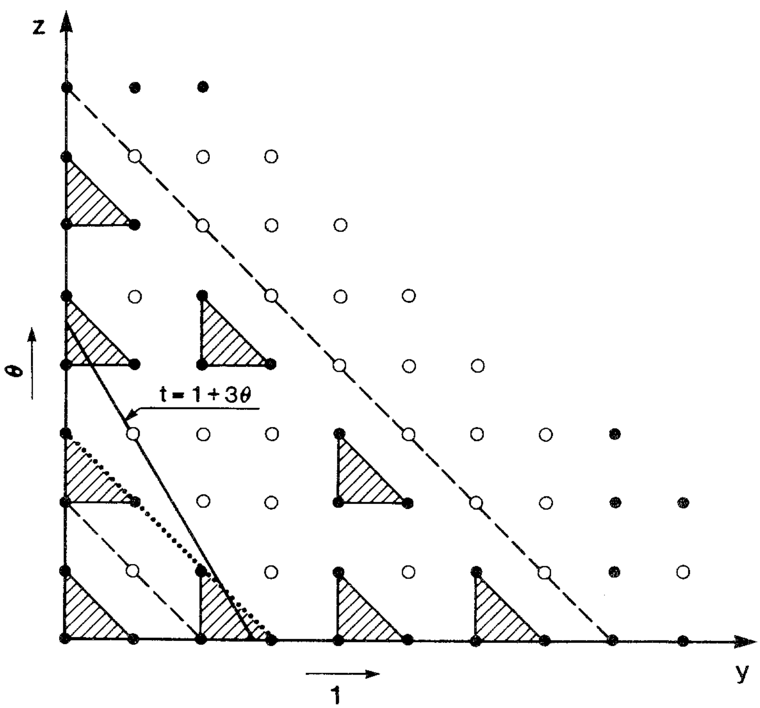


Fig. 1. The delay lattice for Eq. (3.5). ●, jump occurs; ○, jump does not occur. See text for details.

The two dashed lines in the figure are drawn through the pairs of lattice points $(2^k, 0)$, $(0, 2^k\theta)$ for $k=1$ and $k=3$, respectively. The number of jumps occurring before $t=1+3\theta$ can be estimated from above by the number of jumps in the lattice triangle with the dashed line for $k=3$ as its base; it can be estimated from below by the jumps in the triangle corresponding to $k=1$. These two numbers can be computed explicitly in the case at hand.

The computation proceeds by noticing the self-similarity in the pattern of jumps. The lowest-level pattern is given by the small triangles of adjacent jumps (shaded in the figure). The next level is given by the triangle with a dotted line as its base, passing through the point $(2^2-1, 0)$, $(0, (2^2-1)\theta)$. It is easy to show by induction that self-similarity persists to all levels (compare also Wolfram,[6] Fig. 29, where the self-similarity is spatial rather than temporal). As a result the number of jumps $j(k)$ beneath a "dashed" line indexed by k is $j(k)=3^k+2$.

If $\tilde{J}(t)$ is the total number of jumps in (3.5) before time t, one can find k_1, k_2 such that the corresponding "dashed" lines are entirely below and above the isochron $y+z\theta=t$. Thus, by self-similarity,

$$3^{k_1+n}+2 \leqslant \tilde{J}(2^n t) \leqslant 3^{k_2+n}+2 \tag{3.6}$$

which gives immediately

$$\tilde{J}(t)=O(t^{\log_2 3}) \tag{3.7a}$$

The growth of the jump function $J(t)$ itself is accordingly

$$J(t)=O(t^{\log_2 3-1}) \tag{3.7b}$$

Since $\log_2 3-1 \simeq 0.6>0$, the number of jumps per unit time grows, and one can thus say that the solution of (3.5) grows in complexity. This precludes in particular any kind of periodicity or quasiperiodicity. The trivial solution $x(t)\equiv 0$ is the only one which does not exhibit growth in complexity. The jump set, and hence the growth of $J(k)$, is exactly the same when $f(p,q)=p \bigtriangledown q=p\oplus q$ is replaced by $f=p \bigtriangleup q=1\oplus p\oplus q$ in (3.5); the corresponding trivial solution is $x(t)\equiv 1$.

So far, we have addressed only the case when there is a single jump in the initial data, but the proof is valid for any data, by an application of the approximation theorem which will be discussed in the following section. For scalar linear BDEs (3.4) with $\delta \geqslant 3$ incommensurable delays it can be shown by a transparent generalization of the above that

$$J(t)=O(t^{\log_2(\delta+1)}-1) \tag{3.8}$$

This completes the proof of the theorem. ■

3.3. Partially Linear BDEs

Rotating Fig. 1 by $-3\pi/4$, transforming solid circles into ones and open circles into zeros yields Fig. 2.

Pascal's triangles modulo a prime were introduced by Lucas[19] and have been widely studied since. A closely related example will shed some light on the significance of the growth rate of the integral jump function $\tilde{J}(k)$, as the dimension of a self-similar, fractal set.[20–23]

Consider the unit cube $[0, 1]$ in δ dimensions. Divide this into the 2^δ dyadic subcubes and "remove" the interior of the subcube farthest from the origin. Repeating this process *ad infinitum* yields a fractal, generalized Cantor set with Hausdorff dimension $\log_2(\delta+1)$. It is a simple transformation of the "Sierpinski gasket" (Mandelbrot[22]; see also Willson,[24] for linear cellular automata).

We can extend now Theorem 3.1 to nonlinear systems via the GCP (3.3). Given a nonlinear and possibly dissipative BDE system (2.1), define its *index* ν to be the number of terms in its GCP. Factoring out the lowest power of λ in $Q(\lambda)$, one obtains the "characteristic polynomial" of a reversible system. This formal sum of powers of λ is the same as the GCP iff the lowest power of λ present in $Q(\lambda)$ is $\lambda^0 = 1$.

We are ready now to give the final definition of partial linearity, including the case of irrational delays.

Definition 3.2. A system of BDEs (2.20) is *partially linear* if the index of its GCP is high enough, $\nu \geq 3$.

From the discussion at the beginning of Section 3.2 it is clear that the asymptotic behavior of a partially linear system contains the solutions of a linear conservative system with characteristic polynomial as above, having

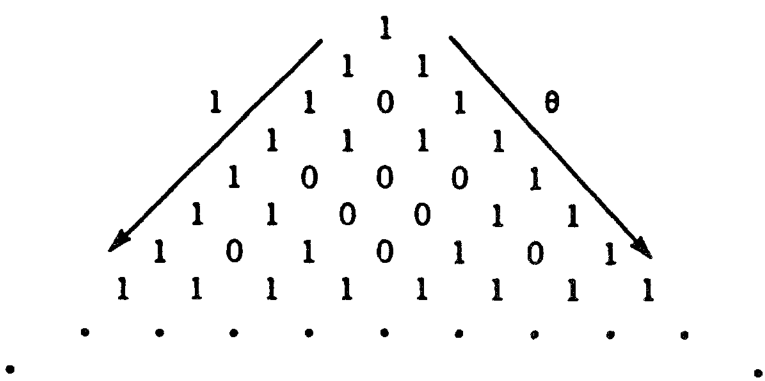


Fig. 2. The delay lattice for Eq. (3.5) as Pascal's triangle (mod 2).

$\nu \geqslant 3$. By Theorem 3.1, this asymptotically embedded system has aperiodic solutions iff sufficiently many of the delays corrresponding to the asymptotically present channels are rationally unrelated. We summarize these results in a theorem.

Theorem 3.2. Partially linear systems of BDEs have aperiodic solutions of asymptotically increasing complexity if the channels of the embedded linear system contain δ rationally independent delays, with $\delta \geqslant 2$.

Remarks. The trivial, asymptotically constant solutions $x_j(t) = 0$ or $x_k(t) = 1$ for $t > T$ are also present. For aperiodic solutions, with $\delta \geqslant \nu - 1$, the asymptotic dimension of the jump set is larger than or equal to $\log_2 \nu$.

The fact that partial linearity, as defined, only gives a sufficient condition for aperiodicity, and that $\log_2 \nu$, for $\delta \geqslant \nu - 1$, is only a lower bound

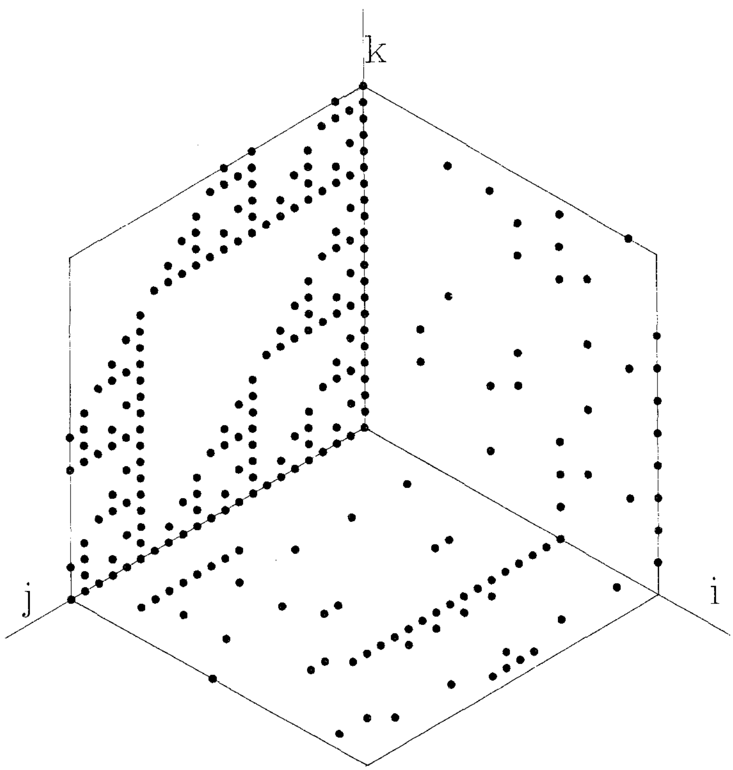


Fig. 3. Delay lattice for Eq. (3.9). Jumps which occur are solid, the others are blank.

for the fractal dimension of the asymptotic jump set can be seen from the following example. Consider the third-order scalar BDE

$$x(t) = [x(t-1) \nabla x(t-\theta)] \wedge \bar{x}(t-\tau) \tag{3.9}$$

with θ, τ and τ/θ irrational, and a single jump in the initial data at t_0, $0 < 1-\theta < 1-\tau < t_0 < 1$.

The delay lattice Γ of this BDE has lattice dimension $\delta = 3$. Its jump set is shown in Fig. 3, as solid circles. The axes j and k correspond to the delays 1 and θ, respectively, while i corresponds to τ.

The jump set in the (j, k) plane indicates self-similar behavior with fractal dimension $\log_2 3$, as in Fig. 1. But the GCP of (3.9) is identically 1, so that $\nu = 1$.

4. APPROXIMATION RESULTS

4.1. The Main Approximation Theorem

In Section 2.1, a metric was introduced into the phase space $X = \mathbb{B}^n[0, 1]$ by Eq. (2.3). Theorem 2.1 stated that the operator $\mathcal{T}_f$, induced by Eqs. (2.1), (2.2) on X, was continuous in this metric. Theorem 2.2 extended this continuity to the dependence on delays, $\mathcal{T}: X \times [0, 1]^{n^2}$.

The purpose of this section is to study the dependence on delays in the large, for solutions of (2.1) on $\mathbb{R}^+ = \{t: t \geqslant 0\}$. For this study we introduce two norms on the solutions of (2.1). In the scalar case, $x: \mathbb{R}^+ \to \mathbb{B}$, these are

$$\|x\|_T = \frac{1}{T} \int_0^T x(t)\, dt \tag{4.1a}$$

where the integration is in the sense of real-valued functions, and

$$\|x\|_\infty = \lim_{T \to \infty} \|x\|_T \tag{4.1b}$$

The limit in (4.1b) always exists for Boolean-valued $x(t)$. The metric induced by (4.1) is clearly (cf. also BDE I)

$$d_{T,\infty}(x, y) = \|x(\cdot) \nabla y(\cdot)\|_{T,\infty} \tag{4.2a,b}$$

with

$$\|x(\cdot) \nabla y(\cdot)\|_T = \frac{1}{T} \int_0^T |x(t) - y(t)|\, dt \tag{4.2c}$$

as in (2.3), where we had $T = 1$.

Consider a sequence of vectors of rational delays $\boldsymbol{\theta}^{(m)}=(\theta_1^{(m)},..., \theta_l^{(m)})$ converging to an arbitrary vector of delays $\boldsymbol{\theta}^{(\infty)}=(\theta_1^{(\infty)},..., \theta_l^{(\infty)})$, where the delays $\theta_1^{(\infty)},..., \theta_l^{(\infty)}$ are all distinct, and $\theta_1^{(\infty)}=1$, say. We denote by $x^{(m)}$ and $x^{(\infty)}$ the associated solutions on $\mathbb{R}^+$, which coincide on $[0, 1]$. The scalar, second-order BDE, $l=2$, is discussed here for simplicity, but the generalization to an $n\times n$ system, with $l\leqslant n^2$, is merely a matter of notation.

Theorem 2.2 stated that $d_2(x^{(m)}, x^{(\infty)})\to 0$ as $\boldsymbol{\theta}^{(m)}\to\boldsymbol{\theta}^{(\infty)}$. The identical argument implies that $d_T(x^{(m)}, x^{(\infty)})\to 0$ for finite, fixed T.

Let $\theta_1^{(m)}=\theta_1^{(\infty)}=1$ and $\theta_2^{(m,\infty)}=\theta_{m,\infty}$ for simplicity, with θ_∞ irrational, and let $x^{(m)}=x^{(\infty)}$ on $[0, 1]$. For sufficiently small $\varepsilon_m=|\theta_m-\theta_\infty|$, the support of $x^{(m)}\nabla x^{(\infty)}$ in $[0, T]$, $T\geqslant 2$, is contained in a small neighborhood of the translates of $\operatorname{supp}\{x^{(m)}\nabla x^{(\infty)}$ restricted to $[1, 2]\}$. More precisely,

$$E_m(T)\equiv d_T(x^{(m)}, x^{(\infty)})\leqslant\varepsilon_m K(T) \tag{4.3a}$$

where $K=K(T; f, \theta_\infty)$ is an unknown, but continuous function of T for $T<\infty$ and for any given BDE defined by the connective f and the delay θ_∞.

We can show that

$$K(T)=O(T^{\alpha+1}) \tag{4.3b}$$

for some $\alpha>0$ (see the proofs of Theorems 2.1 and 3.1). Indeed, the error growth due to increase in the jump function is $O(T^\alpha)$. The dependence on the displacement of a jump, on the other hand, is given by

$$|(t_0+k\theta_m)-(t_0+k\theta_\infty)|\leqslant k\varepsilon_m$$

and k is bounded by $T/\theta_\infty+o(1)$, proving (4.3b). Combining (4.3a) and (4.3b) yields

$$E_m(T)\leqslant O(T^{\alpha+1}q_m^{-1}) \tag{4.4}$$

where $\theta_m=p_m/q_m$ is in reduced form, $\varepsilon_m=O(q_m^{-1})$, and q_m is taken to increase monotonically as $\theta_m\to\theta_\infty$ (see Appendix B of AM).

Hence, for prescribed accuracy $E_m(T)\leqslant\eta$, one obtains from (4.4)

$$T=O(q_m^{1/(1+\alpha)}) \tag{4.5}$$

In other words, the approximation $x^{(m)}(t)=x(t; p_m/q_m)$ to $x^{(\infty)}(t)=x(t; \theta_\infty)$ retains its accuracy $O(\eta)$ for longer and longer times as $\theta_m=p_m/q_m\to\theta_\infty$. Furthermore, T can be chosen to grow more slowly than (4.5), in such a way that $E_m(T)\to 0$.

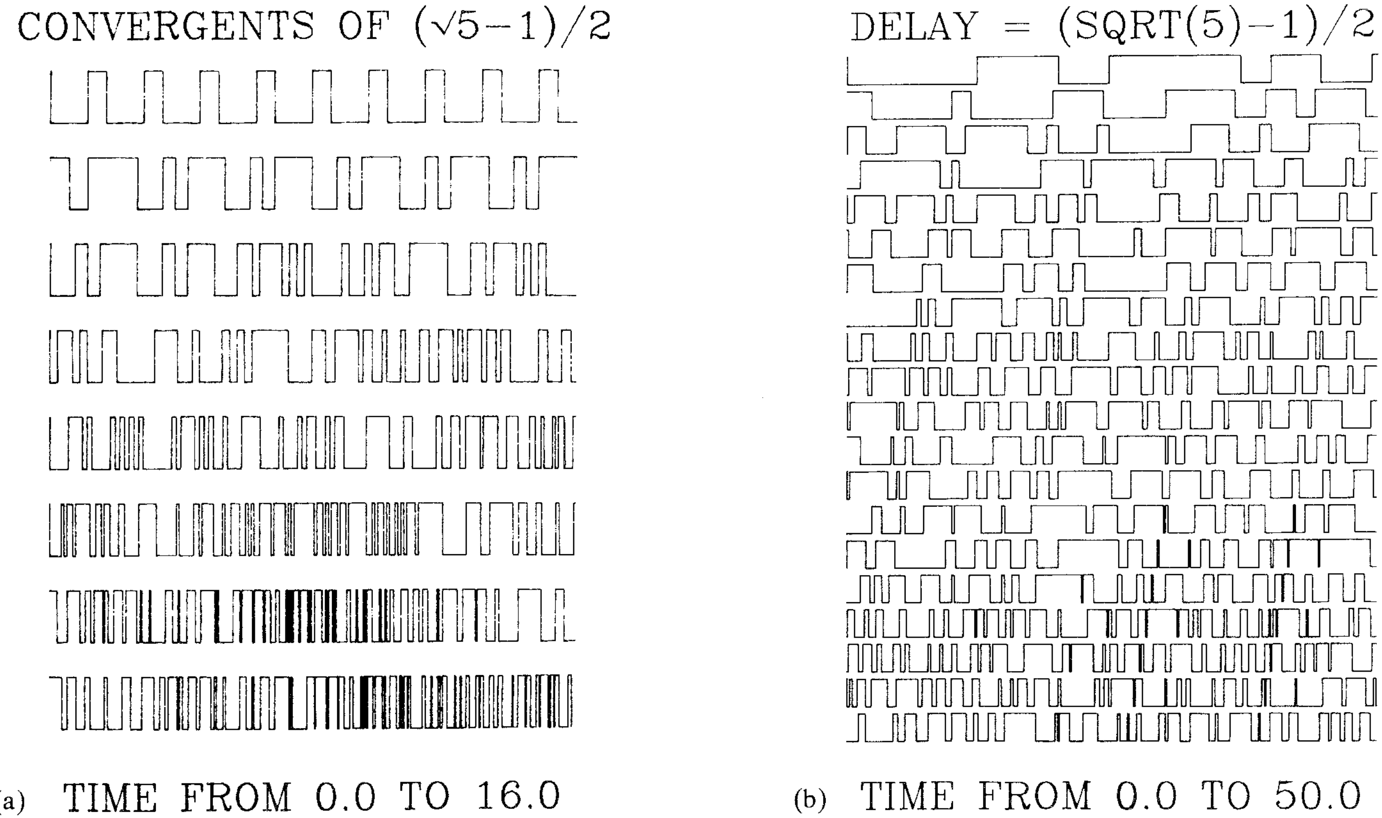
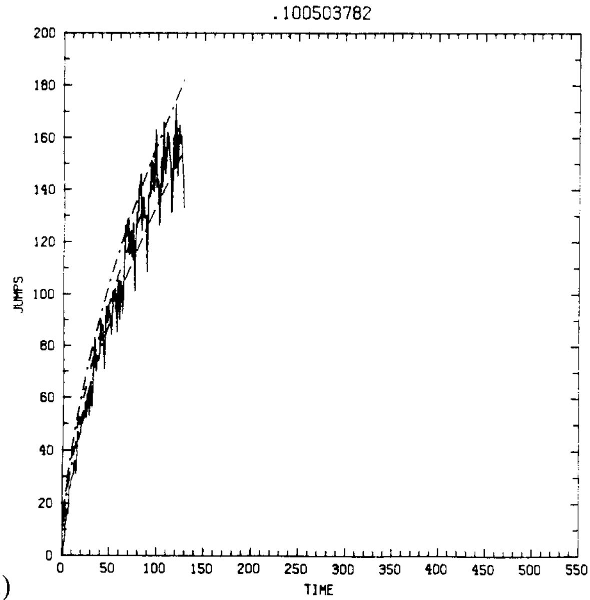
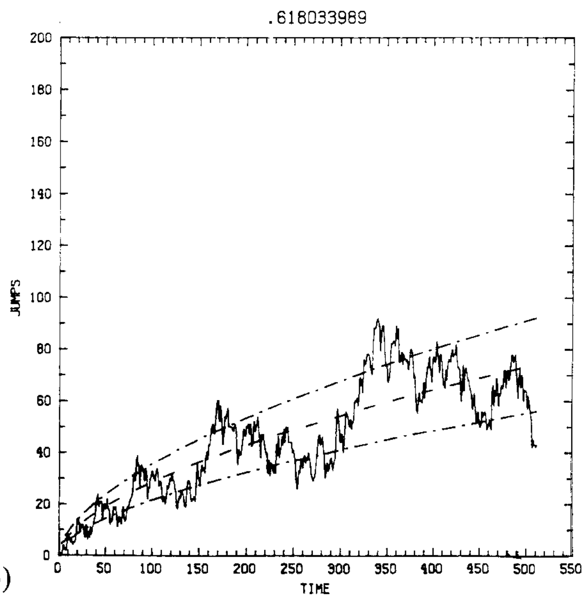


Fig. 4. Solutions of Eq. (3.5) for $\theta = (\sqrt{5}-1)/2$. (a) Approximation by convergents $\theta_1 = 1/2$, $\theta_2 = 2/3$, $\theta_3 = 3/5$, $\theta_4 = 5/8$, $\theta_5 = 8/13$, $\theta_6 = 13/21$, $\theta_7 = 21/34$, and $\theta_8 = 34/55$, for $T = 16$; (b) solution for exact θ and $T = 60$.

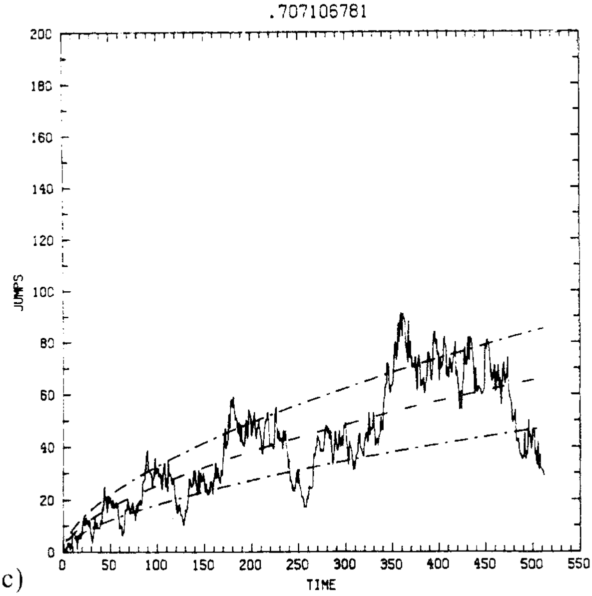


(a)

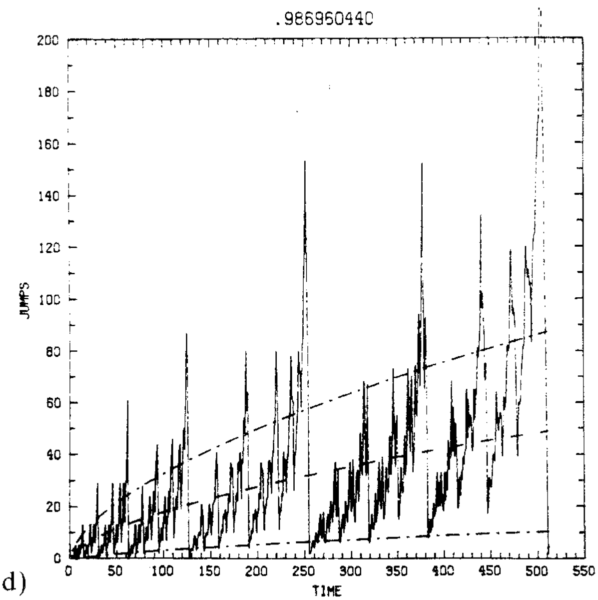


(b)

Fig. 5. Jump function $J(t)$ of Eq. (3.5) for $0<\theta<1$. (a) $\theta=1/\sqrt{99}$; (b) $\theta=(\sqrt{5}-1)/2$; (c) $\theta=1/\sqrt{2}$; and (d) $\theta=\pi^2/10$. Notice increased intermittency as θ increases from 0 to 1.



(c)



(d)

Fig. 5 (*continued*)

Specifically, let $T_m = T(q_m)$ grow asymptotically like

$$T_m = O(q_m^{1/P(\alpha+1)}) \tag{4.6a}$$

where P is a fixed positive integer, $P \geqslant 2$. Then, by (4.4),

$$E_m(T_m) = O(q_m^{-1+1/P}) \tag{4.6b}$$

This argument is easily extended to prove the following.

Theorem 4.1 (main approximation theorem). All solutions to systems of BDEs can be approximated by the periodic solutions of a nearby system with rational delays only. The accuracy of the approximation E_m is given by (4.6b) up to a time T_m given by (4.6a).

The results of a sequence of approximations by continued fractions (see AM, Section 4.1 and Appendix B) are shown in Fig. 4a for Eq. (3.5) and θ equal to the golden ratio. Figure 4b shows the exact solution over a longer time interval.

4.2. Self-Similarity and Intermittency of Aperiodic Solutions

Figure 5 displays the jump function $J(t)$ for the same equation, a single jump in the initial data, and various irrational values of the delay θ. The self-similarity of $J(t)$ apparent in Figs. 5a–d (solid line) is striking. The second striking fact about Fig. 5, and less expected, is the increase in intermittency as θ increases from 0 to 1. Figure 6 shows the actual solution for the last value of θ (Fig. 5d), $\theta = \pi^2/10$.

To explain the variation in behavior of $J_f(t; \theta)$ as θ changes, recall that by Eq. (3.7), for all irrational values of θ,

$$J(t; \theta) \sim K(\theta)\, t^{\log_2 3 - 1} \tag{4.7}$$

Figures 5a–d show that K is a strong function of θ, being largest when θ is nearly 0 and smallest as θ approaches 1.

The constant K in Eq. (4.7) characterizes the *mean* asymptotic growth behavior of the jump function $J(t)$. Figure 5 also shows that, while K decreases with θ, the *relative variability* of $J(t)$ increases with θ. This is evidenced by the fact that the ratio of the distance between the upper and the lower "envelopes" of $J(t)$ (dash-dotted lines in the figure) to the ordinate of the "mean" (dashed) increases from panel to panel in the figure. Since the mean growth is smaller for larger θ, this increasing ratio is related to the very intermittent character of solutions for high θ (Fig. 6).

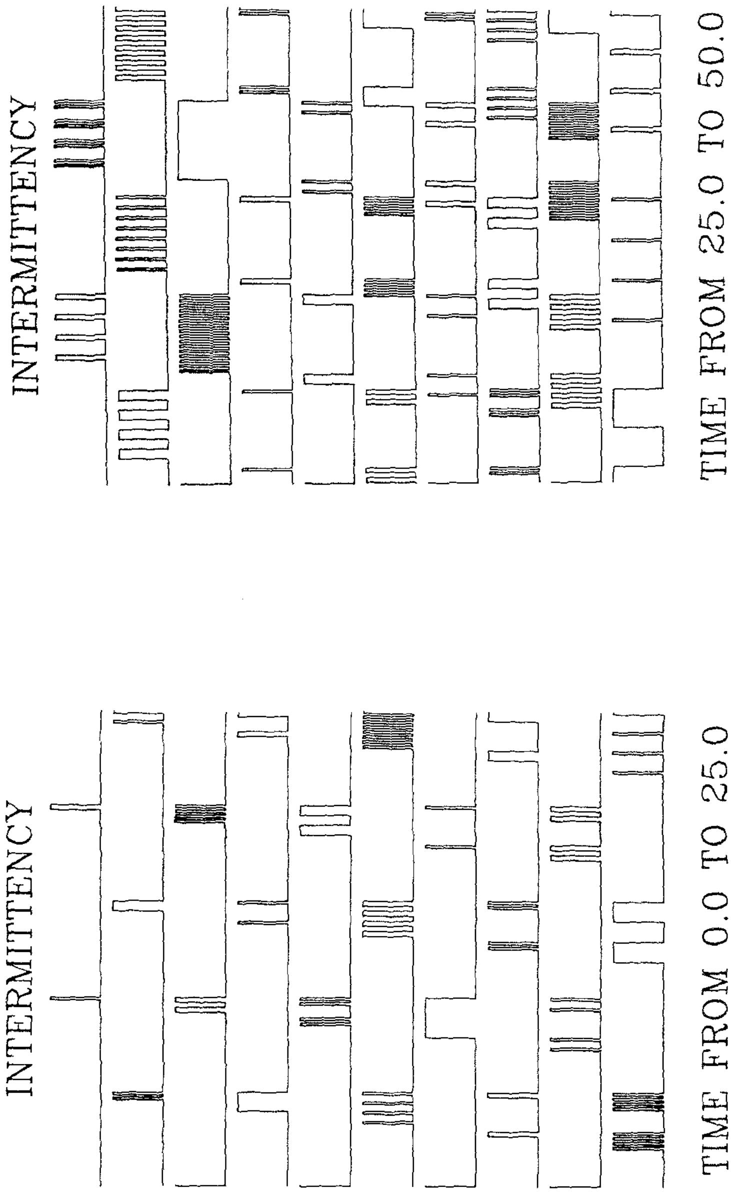


Fig. 6. Solution $x(t)$ of Eq. (3.5) for $\theta = \pi^2/10$. Notice self-similarity and strong intermittency. (a) $0 \leq t \leq 25$; (b) $25 \leq t \leq 50$.

The function $J_f(t;\theta)/K_0 t^\alpha$ for the connective f of Eq. (3.5), $\alpha = \log_2 3 - 1$, $\theta = 1/\sqrt{2}$ (compare Fig. 5c) and $K_0 = 1.814$ is plotted in Fig. 7a. It is clear that this function oscillates, with no apparent decrease in the amplitude of the oscillation.

The same function is plotted in Fig. 7b with $\log_2 t$, rather than t itself, on the abscissa. This plot shows that $J(t)/t^\alpha$ is asymptotically $\log_2 t$ periodic in the coarsest pattern, but with an increase of fine structure from coarse period to coarse period. This increase is to be expected from the knowledge of the self-similar, fractal character of the jump lattice.

Figure 7b suggests that we consider $g(t) = J(2^t)$ which, in the limit, is periodic in the usual sense. For this function,

$$\lim_{t\to\infty} \frac{1}{t} \int_1^t g(s)\, ds = \text{const.} \cong 2$$

and this limit suggests in turn a reasonable quantification for the mean behavior of $J(t)$. We shall thus define

$$K(\theta) = \lim_{t\to\infty} \int_1^t J(2^s)\, 2^{-\alpha s}\, ds \tag{4.8a}$$

$$M_p(\theta) = \lim_{t\to\infty} \int_1^t [J(2^s)\, 2^{-\alpha s} - K(\theta)]^p\, ds, \quad p \geqslant 2 \tag{4.8b}$$

for all aperiodic solutions of growing complexity.

Theorem 4.2. The mean asymptotic behavior $\bar{J}(t)$ of the jump function $J(t;\theta)$ for the linear, conservative BDE (3.5) is given by

$$\bar{J}(t) = K(\theta)\, t^\alpha \tag{4.9}$$

where $\alpha = \log_2 3 - 1$ and $K(\theta)$ is a monotone decreasing function of θ irrational on $0 < \theta < 1$.

Remark. It can be shown more generally for any system (2.1) of BDEs with rationally independent delays $\boldsymbol{\theta} = (\theta_1, ..., \theta_l)$, $0 < \theta_1 < \cdots < \theta_l = 1$ that $K_f(\boldsymbol{\theta})$ is a monotone decreasing function function of $\prod_{i=1}^{l} \theta_i$.

Proof. See AM.

Returning to Fig. 5, we can now state the exact definition of the dashed "mean growth" and the dash-dotted "envelope" lines: they are the plots of $\bar{J}(t;\theta) = K(\theta)\, t^\alpha$ and of $\bar{J} \pm M_2^{1/2}(\theta)\, t^\alpha$, respectively.

We measure solution *intermittency* for BDEs by the relative variability of their jump functions. Thus, given the scaling of (4.8), *relative variability*, and therewith intermittency, is defined as

$$\rho(\theta) = M_2(\theta)/K(\theta) \tag{4.10}$$

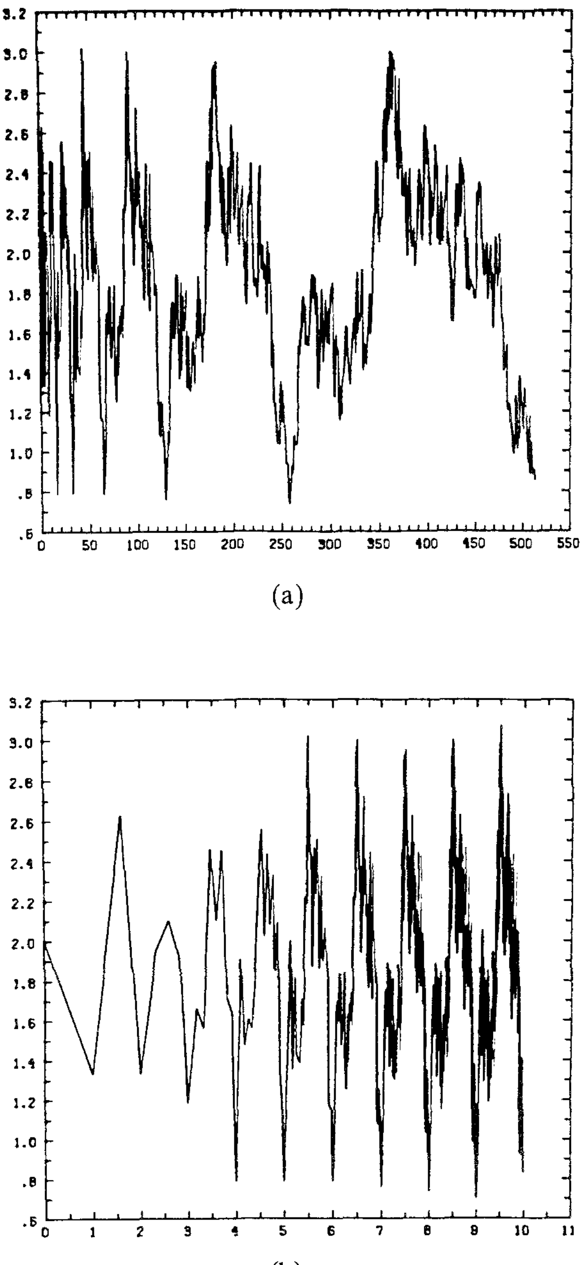


Fig. 7. Scaled jump function $J(t)$ for Eq. (3.5) and $\theta = 1/\sqrt{2}$: (a) vertical normalization $J(t)/K_0 t^{\alpha}$; (b) same vertical scaling and logarithmic scaling in the horizontal.

Figure 8 shows a plot of $\rho(\theta)$. The general increase of ρ with θ for $0.4 < \theta < 0.9$ is obvious. It is also clear from the peaks superimposed on this general growth of $\rho(\theta)$ that the situation is considerably more complicated than a monotone increase with θ.

The difficulty, trivially resolved for $K(\theta)$, is created here by the effect of *resonances*, due to the presence of rational delays in the interval. For θ rational, $\alpha = 0$ and $\rho(\theta)$ is of the indeterminate form 0/0.

The points in Fig. 8, however, were obtained by using an efficient BDE solver (see Appendix A of AM) which, when given a rational delay, produces the jump function corresponding to an irrational delay close by. This results in the peaks appearing in the figure. The width of each peak is related to the strength of the resonance caused by the given rational delay.

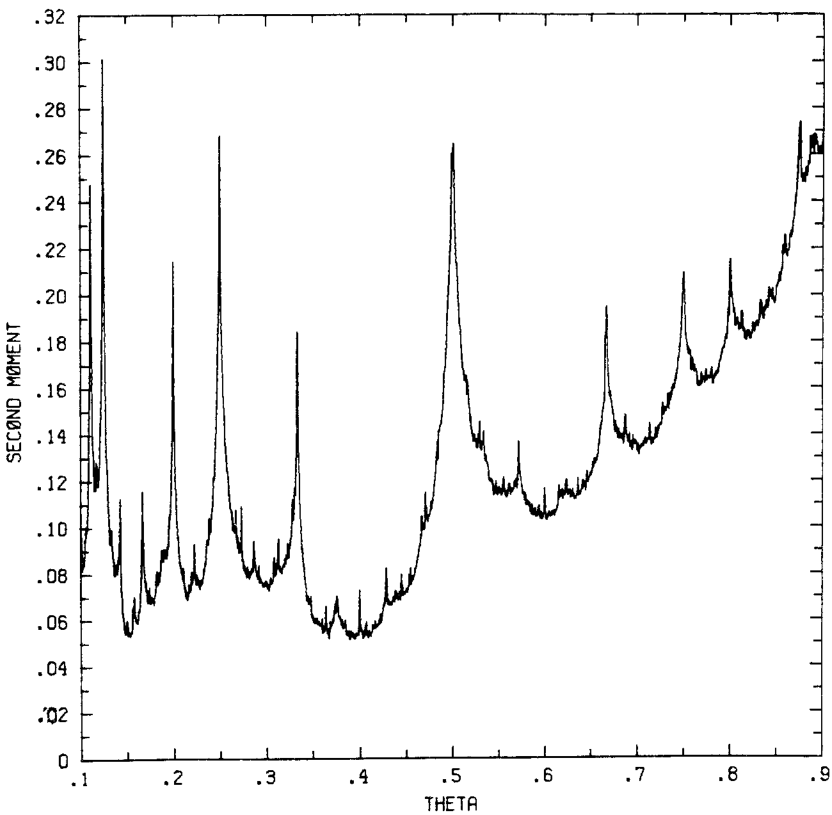


Fig. 8. Relative variability $\rho = M_2/K$ for Eq. (3.5) as a function of delay θ. Notice general increase with θ and resonant peaks near simple rational values of θ.

The height of the peaks, on the other hand, is more difficult to interpret. Owing to the extreme narrowness of the highest portion of the peaks, finite sampling in θ can miss the "top" of the peak by a significant amount. By the same token, the level of the entire computed curve might be lower than in reality by the amount indicated by the very smallest peaks visible, so that these could be actually spurious.

Even so, further numerical investigations suggest that a considerable amount of the fine structure apparent in Fig. 8 is real. This structure indicates that $\rho(\theta)$, and hence intermittency, increases drastically for irrational delays well approximated by rationals.

The qualitative explanation of this fact is simple. We recall from Section 4.1 that over some fraction of the period of the solution generated by the rational delay, this solution approximates well that generated by the irrational delay. The jumps of the irrational solution which do not correspond to jumps in the rational one are due to a failure of the former jumps to coincide and cancel, as their rational cousins do. Each pair of such non-canceling but very close jumps in the irrational solution propagate independently and undisturbed, the two series of progeny jumps shadowing each other closely (see also BDE I, Section 4).

This process repeats on smaller and smaller scales as cancellations with higher integer coefficients occur in the rational solution. That is, after longer and longer times, more lattice points lie on the same rational isochron in Fig. 1. It appears that the simpler the approximating rational delay, i.e., the lower its denominator, the faster the process happens. This provides a plausible explanation for the width and height of the peaks in Fig. 8 apparently centered at certain simple rational delays ($\theta = 1/2$, $1/3$, $1/4$, $1/5$).

4.3. Resonances and Scaling

We turn now to a more quantitative analysis of partially linear BDEs. In certain exceptional cases, when resonance phenomena occur, there are no aperiodic solutions for any delays $\mathbf{t}$, or the growth of solutions in time is slower than otherwise expected.

Definition 4.1. Given delays $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)^T$ of the homogeneous linear scalar BDE (3.4) associated with the partially linear system (2.20), a *resonance* is defined by an inner product relation

$$\mathbf{a}^T\boldsymbol{\theta} \equiv \sum_1^m a_k \theta_k = 0 \tag{4.11a}$$

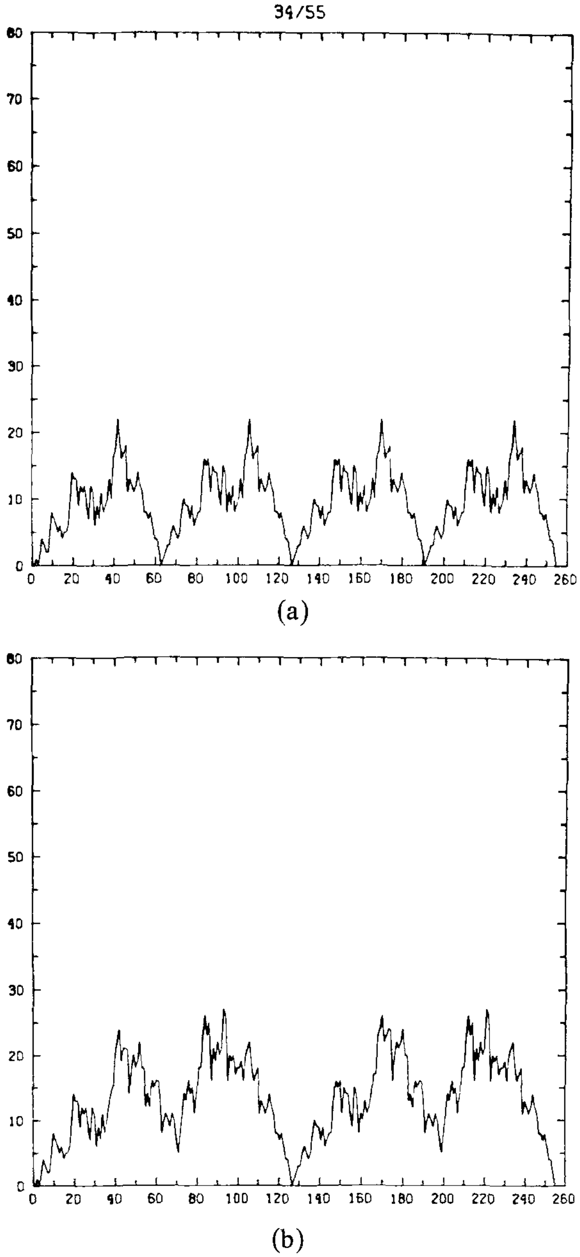
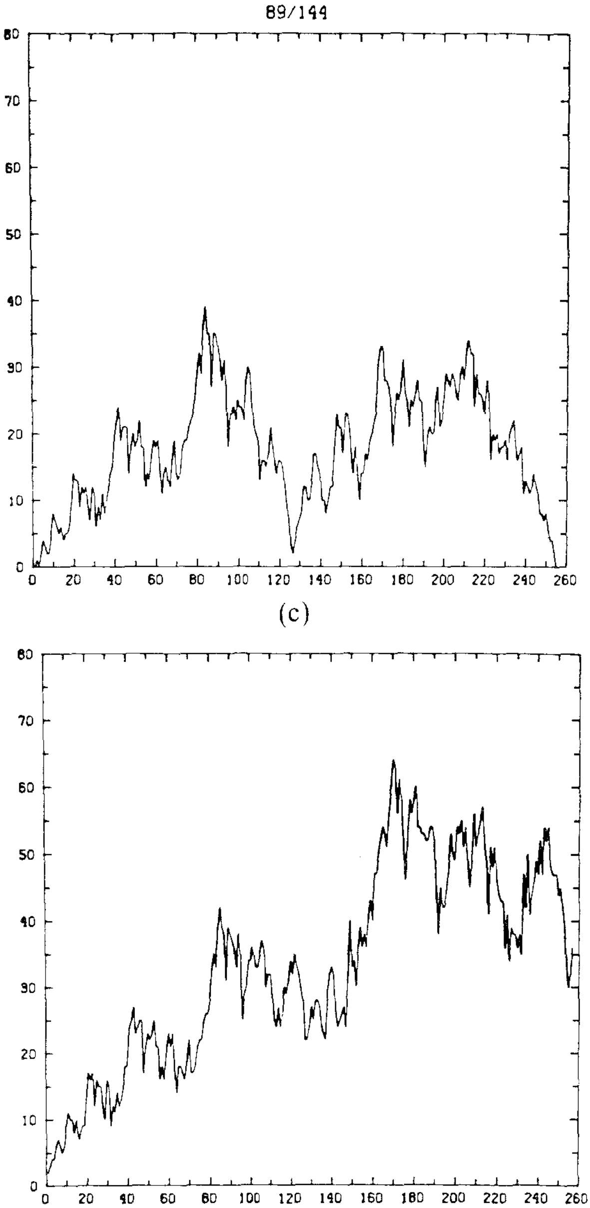


Fig. 9. Jump function of Eq. (3.5) for $\theta^{(\infty)} = (\sqrt{5} - 1)/2$ (panel 9d: see also Fig. 5b) and continued fraction approximants with $q = 55$, 89, and 144 (panels 9a, 9b, and 9c, respectively).



(c)

(d)

Fig. 9 (*continued*)

where a_k are integers and

$$|\mathbf{a}| \equiv \sum_{1}^{m} |a_k| \geqslant 2 \tag{4.11b}$$

is the *order* of the resonance.

The delays $\boldsymbol{\theta}$ are given by linear combinations of the delays $\mathbf{t} = (t_1, \ldots, t_N)^T$ of the original system,

$$\boldsymbol{\theta} = L\mathbf{t} \tag{4.12}$$

where $\boldsymbol{\theta}$ and $\mathbf{t} = (t_l)$ are column vectors, $(\cdot)^T$ is the transpose of a vector $(\cdot)$, L is a matrix with nonnegative, integer entries, the single index l corresponds to a linear ordering of the pairs (i, j) in (2.20) and $N \leqslant n^2$ is the actual number of distinct delays appearing in the system. The rank d of L gives the maximum number of components of $\boldsymbol{\theta}$ which can be rationally independent, regardless of the values of $\mathbf{t}$.

If $d \leqslant 1$, then no aperiodic solutions can exist. Since L depends only on the connective $\mathbf{f}$, we call this first type of degeneracy *system resonance*. More generally, the dimension δ of the delay lattice of the scalar BDE, i.e., the number of its rationally independent delays, is at most d, and the highest dimension of its jump set is given by $\alpha + 1 = \log_2 \min\{\delta + 1, \nu\}$. Thus, when $d = \nu - 1$ we say that L has *maximal rank*, and any value of $d < \nu - 1$ corresponds to system resonance.

A second type of resonance occurs when rational relations exist among the delays $\{t_{ij}\}$ of the original system. We shall call it *delay resonance*. For such relations to affect the associated scalar BDE, they must occur among the delays present in the partially linear channels.

Suppose $\boldsymbol{\theta} = L\mathbf{t}$ has resonance $\mathbf{a}$. Then

$$\mathbf{a}^T\boldsymbol{\theta} = \mathbf{a}^T(L\mathbf{t}) = (\mathbf{a}^T L)\mathbf{t} = 0 \tag{4.13}$$

and $\mathbf{t}$ has resonance $\mathbf{b} = L^T\mathbf{a}$, since L has integer elements. This is a genuine resonance, according to (4.11b), only if $|L^T\mathbf{a}| \geqslant 2$. But $L^T\mathbf{a} = 0$ iff $\mathbf{a}$ is not in the range of L. Hence delay resonance $\mathbf{b}$ in (2.20) will have caused the observed resonance $\mathbf{a}$ in (3.4) only if $\mathbf{a}$ has a nontrivial component in the range of L. Otherwise the resonance $\mathbf{a}$ has to come from system resonance, i.e., from d being less than maximal.

We summarize these results in the following theorem.

Theorem 4.3. If the rank of L is maximal and no delay resonance occurs, the asymptotic growth of complexity in solutions of Eq. (3.4), and hence of system (3.2), will be maximal, with scaling exponent of the jump function $\alpha = \log_2 \nu - 1$. On the other hand, if $d < \nu - 1$, then $\alpha \leqslant \log_2(d+1) - 1$. The effect of a resonance $\mathbf{b}$ among delays $\mathbf{t}$ of (2.20) on α depends on the actual presence of the delays in (3.2) and on the components of $\mathbf{b}$ lying in the range of L^T.

Remarks. (1) Computationally, delays $\boldsymbol{\theta}$ with a resonance of high order $|\mathbf{a}|$ will behave like nonresonant delays for a very long time.

(2) Over any finite time, solutions to BDEs are easier to compute than solutions of ordinary differential or smooth functional differential equations. But the asymptotic behavior of a dynamical system, whether continuous or discrete, requires an amount of computation dictated essentially by its *complexity* (see Appendix A of AM), rather than by the local rate of error growth.

A simple example illustrating the two types of resonance discussed is given in AM, Eqs. (4.17), (4.18).

It can be shown, using Dirichlet's theorem,[25] that all sufficiently good rational approximations admit all of the resonances of the delays they approximate. This means that for a reasonable selection of the approximating delays, we will observe any resonance of the exact solution $\mathbf{x}^{(\infty)}$ in the approximating solution $\mathbf{x}^{(k)}$.

The length of time needed in order to observe a resonance of the exact solution in the approximation depends on the approximating delays and is estimated in Section 4.3 of AM. We merely state here that, when a resonance occurs, the scaling exponent α of the jump function decreases. Hence a sequence of resonances will manifest itself in the solution as successive decreases in the rate of growth of the jump function. These decreases occur in sequence, according to the order of the resonance. As a result, the actual time at which the growth associated with a given resonance becomes apparent depends on the previously manifested, lower-order resonances.

One last question concerning resonances and approximation refers to determining, for given delays $\boldsymbol{\theta}^{(\infty)}$, whether sufficiently close approximants $\boldsymbol{\theta}^{(k)}$ [cf. (4.4)] admit spurious resonances. This problem can be treated for almost all (in the sense of Lebesgue), but not all, choices of delays.

We refer to delays $\boldsymbol{\theta}$ as being of type (C, ν) (in analogy with Arnold,[26] Section 24, although the meaning of resonance here is different) if

$$|\mathbf{a}^T\boldsymbol{\theta}| \geqslant C\,|\mathbf{a}^T|^{-\nu} \tag{4.14}$$

for all resonances $\mathbf{a}^T$, with $C > 0$ and $\nu > 0$ fixed. For $\nu > m+1$, almost all delays $\boldsymbol{\theta}$ are (C, ν) for some $C > 0$. If $\boldsymbol{\theta}$ is of type (C, ν), then it admits no resonances, but the converse is not true, as shown by the example of Liouville numbers (see Appendix B of AM).

If $\boldsymbol{\theta}^{(\infty)} = \boldsymbol{\theta}$ is (C, v) and an approximant $\boldsymbol{\theta}^{(k)} = \mathbf{p}^{(k)}/q$ satisfies

$$\left|\theta_k - \frac{p_k}{q}\right| < \frac{1}{q^{1+1/m}}, \qquad 1 \leqslant i \leqslant m$$

then the condition

$$q^{1+1/m} > |\mathbf{a}^T|^{v+1/C}$$

or

$$q > (|\mathbf{a}^T|^{v+1/C})^{m/(m+1)}$$

is sufficient in order to avoid spurious resonances of order $|\mathbf{a}|$ or below. Notice also that if the system delays $\mathbf{t}$ are (C, v) and L has maximal rank, then the scalar delays $\boldsymbol{\theta}$ are (C', v) for some $C' > 0$ which depends on L.

5. PERIODIC SOLUTIONS

5.1. The Period Function

Systems of BDEs which have only rational delays, and therefore only periodic solutions (Section 2.2), are dense in the space of all systems of BDEs. On the other hand, for almost all linear and partially linear systems only the trivial solutions are periodic, so that almost all solutions are aperiodic (Sections 3 and 4.3). Furthermore, the approximation results of Section 4.1 indicate that systems with aperiodic solutions are well approximated, up to a certain time, by systems with slightly perturbed delays and periodic solutions.

This raises a number of questions about the dependence of *maximum period length* π on the connective $\mathbf{f}$ and on the delays $\mathbf{t}$. For linear or partially linear systems, $\pi_f(\mathbf{t}) \to +\infty$ as some delays t_{ij} become irrational. For systems which are not partially linear, and which are not asymptotically linear (cf. Sections 3.3 and 6), we suspect that the function $\pi(\mathbf{t})$ is bounded or that the solutions are quasiperiodic.

Theorem 5.1. The period function $\pi(\mathbf{t})$ is a lower semicontinuous function of the delays $\mathbf{t}$ for any BDE (2.1), in any $\mathbf{t}$ interval in which it is bounded away from zero.

Proof. See AM.

Remark. Theorem 5.1 can be weakened so that it applies to all $\mathbf{t}$ intervals, by stating that the function $\pi^*(\mathbf{t}) = \pi(\mathbf{t}) + 1/\pi(\mathbf{t})$ is lower semicontinuous, without further qualification.

By the definition of semicontinuity,

$$\pi(\mathbf{t}) = \sup_{\varepsilon > 0} \inf_{\|\mathbf{t} - \mathbf{t}'\| < \varepsilon} \pi(\mathbf{t}') \tag{5.1}$$

In fact, our approximation results provide more information than just (5.1). A semicontinuous function in general can have jump discontinuities. These can only occur in the period function if the solution corresponding to the delay vector $\mathbf{t}^{(0)}$ at which the jump in $\pi(\mathbf{t})$ occurs actually vanishes, or is identically equal to 1.

Hence, it is not hard to prove (see AM):

Theorem 5.2. In a neighborhood of a delay vector $\mathbf{t}^{(0)}$ in which the period function $\pi(\mathbf{t})$ is bounded away from infinity and from zero, $\pi(\mathbf{t})$ is continuous.

Various special cases of discontinuities are discussed in AM.

5.2. Upper Bounds for Period Lengths

For simplicity, we consider a scalar second-order BDE [not necessarily (3.5)] with $\theta_1 = 1$ and $0 < \theta_2 < 1$. Let $\theta_2 = \theta = p/q$ with p and q relatively prime. Decompose 2^q into all possible sums of positive integers, and let $M(q)$ be the supremum over all decompositions of the least common multiples (l.c.m.) of the summands in each decomposition. If the BDE has $k = 1$ or $k = 2$ constant solutions, $M(q)$ will be defined with respect to $2^q - k$ rather than 2^q.

Theorem 5.3.

$$\pi(p/q) \leqslant M(q)/q. \tag{5.2}$$

Remark. In the general case of a system with $t_l = p_l/q_l$, $q = \text{l.c.m.}\{q_l\}$ in (5.2).

Proof. This is essentially a refinement of the proof of Theorem 2.3. Let $\tau(s)$ be a q-tuple of 0's and 1's defined by

$$\tau(s) = [x(s), x(s + 1/q), \ldots, x(s + (q-1)/q)] \tag{5.3}$$

The initial data for the BDE with $\theta = p/q$ can be represented as a one-parameter family of such q-tuples, where $s \in [0, 1/q)$ is the parameter.

This representation of the initial data induces a representation of the solutions, as

$$x(t) = x([t] + s) = \tau(n + s) \tag{5.4}$$

where $[t]=n$ is the greatest integer less than or equal to t, and s is the fractional part of t. The crucial observation is that

$$\tau(s+1)=\phi(\tau(s)) \tag{5.5a}$$

where ϕ does not depend on s. Hence each tuple $\tau(s)$ propagates in the solution of the BDE independently of the other tuples.

Continuation of the solution for this rational θ corresponds in fact to matching the first $q-1$ components of each q-tuple with the last $q-1$ components of its predecessor:

$$\tau(s+1/q)=\Phi(\tau(s)) \tag{5.5b}$$

Then $\phi=\Phi^q$ and Φ is a representation of $\mathscr{T}^{1/q}$ [compare Eq. (2.5) and Lemma 2.1].

Each q-tuple $x(t)=\tau(s)$, $[t]=0$, in the initial data generates a periodic subsolution according to Eq. (5.5). The length $l(\tau)$ of the period is determined from the first repetition of the tuple in the subsolution, e.g., $\tau(s+r/q)=\tau(s)$, for some smallest integer r.

The subsolutions can at most exhaust the entire phase space, which has 2^q-k points. Let $2^q-k=\sum_i n_i$ be a particular decomposition into arbitrary positive integers, and the initial data contain q-tuples $\tau_i=\tau(s_i)$ which lead to words of lengths $n_i=l(\tau_i)$. The solution in this worst case has a period not longer than $\text{l.c.m.}\{n_i\}/q$. This proves the theorem. ■

In general $M(q)$ is much larger than 2^q-k. For linear equations, however, one can show that $\pi(q)\leqslant(2^q-k)/q$ (AM, Theorem 5.4). The case with rational delays is reducible to a well known chapter in the theory of shift registers[16] and is also analogous to certain classes of cellular automata.[24,27]

A better understanding of period length in the nonlinear case requires the study of dissipative systems, to which we turn presently.

6. DISSIPATIVE BDEs

Throughout much of the last three sections, the study of jump propagation by a system's linear part was the foundation of our theory. In order to discuss the effects of dissipativity and nonlinearity, a change in perspective is required. In the present section, we will develop some alternative techniques and apply them to the wide class of dissipative BDEs.

6.1. Normal Forms of Connectives

It simplifies the exposition to restrict our attention to the single, nth-order scalar BDE

$$x(t) = f(x(t-\theta_1),\ldots, x(t-\theta_n)) \tag{6.1}$$

where $0 < \theta_n < \cdots < \theta_1 = 1$, and f is any function $f\colon \mathbb{B}^n \to \mathbb{B}$. Since we do not impose further restriction on f, we shall need certain well-known facts from Boolean algebra in order to represent its structure in simplest, standard terms.[15,28]

Let

$$u_i = x(t-\theta_i), \qquad i = 1,\ldots, n$$

and $u = x(t)$.

We shall write in the present section pq for $p \wedge q$ and $p+q$ for $p \vee q$. With this notation, common in switching and automata theory,

$$p + q = p \oplus q \oplus pq$$

where $p \oplus q = p \bigtriangledown q$ is addition over $\mathbb{Z}_2$, as before. Then $f(\mathbf{u})$ of (6.1) can be written in the form (see AM for details)

$$f(\mathbf{u}) = \sum_{f(\mathbf{v})=1} \left[\prod_{k=1}^{m(\mathbf{v})} u_{l_k}(\mathbf{v}) \prod_{k=m(\mathbf{v})+1}^{n} \bar{u}_{l_k}(\mathbf{v}) \right] \tag{6.2}$$

where $\mathbf{v}$ is a dummy variable ranging over $\mathbb{Z}_2^n$. Equation (6.2) is called the *disjunctive normal form* (DNF) of $f(\mathbf{u})$, with $\bar{p} = 1 - p$. By de Morgan's rule we also have the *conjunctive normal form* (CNF),

$$f(\mathbf{u}) = \prod_{f(\mathbf{v})=0} \left[\sum_{k=1}^{m(\mathbf{v})} \bar{u}_{l_k}(\mathbf{v}) + \sum_{k=m(\mathbf{v})+1}^{n} u_{l_k}(\mathbf{v}) \right] \tag{6.3}$$

We write $p \to q$ ("p implies q") to indicate that q must be *true*, $q = 1$, whenever p is:

$$(p \to q) = (\bar{p} + q)$$

Note that $f \to g$ iff g is a product of factors of (6.3), and $g \to f$ iff g is a sum of terms of (6.2). For this reason, a product of factors of (6.3) is called an *implicate* of f and a sum of terms of (6.2) is called an *implicant* of f. Trivially, $f \to 1$ and $0 \to f$.

Given these conventions, there are two equivalent logical descriptions of (6.1):

$$
\begin{array}{lcl}
x(t) \to \psi_1, & & \phi_1 \to x(t) \\
x(t) \to \psi_2, & \text{or} & \phi_2 \to x(t) \\
\vdots & & \vdots \\
x(t) \to \psi_r, & & \phi_s \to x(t)
\end{array}
\tag{6.4a,b}
$$

where $\psi_1,..., \psi_r$ are all the implicates of $x(t)$ according to f and (6.3), and $\phi_1,..., \phi_s$ are all the implicants of $x(t)$ according to f and (6.2). A simplification is possible iff some of the implicates or implicants in (6.4) may be safely ignored. At this stage, the particular nature of Boolean delay equations may provide additional information, since the expressions for the implicates and implicants involve the same variable, $x(t')$, at different moments in its time evolution, $t' = t - \theta_j$, $j = 1,..., n$.

6.2. Asymptotic Simplification of BDEs

A good setting for these questions is to consider an irredundant subset of implications (6.4) as part of a grammar G generated by the connective f of (6.1). This grammar governs a formal language whose words make up the solutions of (6.1); see for details AM (Section 6.2) and Hopcroft and Ullman[29] (Chapter 2). In the sequel, we refer for brevity to the original set of implications, $\mathscr{G}$, without creating undue confusion, rather than to the grammar G.

Since the original set $\mathscr{G}$ of implications is irredundant, the removal of any on of them would alter the connective. However, if for some time t_0 the situation arises that an implication β of $\mathscr{G}$ can be derived from $\mathscr{G} - \{\beta\}$ augmented by β restricted to times less than t_0, then the implication β has become redundant after some time in the evolution of the equation. From Sections 2.3 and 2.4 we suspect that such a situation might arise after all transients have died out.

This means that after sufficient time, the evolution of the solution to the original BDE is governed by a BDE with the implication β absent from its connective, but which still contains other implications of $\mathscr{G}$. Since $\mathscr{G}$ is irredundant, this new connective is not the same connective, i.e., the asymptotic behavior reduces to a simpler system. Removal of implications corresponds to elementary modifications of the appropriate normal form, hence a connective exists which governs this asymptotic evolution.

Another interesting situation arises when for some integer combination of delays π, the variable $x(t-\pi)$ is implied by $x(t)$ and $\bar{x}(t-\pi)$ is implied by $\bar{x}(t)$, i.e., $x(t) \to x(t-\pi)$ and $x(t-\pi) \to x(t)$. This states simply that a period can be determined by the technique we have sketched. There is a converse to this statement, namely, that any determination of the period by other means yields two useful implications.

The following theorems illustrate the two situations above, the asymptotic and the periodic one.

Theorem 6.1. Solutions to Eq. (6.1), where f carries implications $\alpha: x(t) \to \mathbf{x}(t-\theta_\alpha)$ and $\beta: x(t) \to \bar{x}(t-\theta_\beta)$, solve eventually an equation which does not contain β.

Proof. We want to show that β is redundant for $t > t_0$, where t_0 is the maximum length of transients. The following diagram contains the proof, using ( )* to indicate contraposition, and ( )$_-$ to indicate restriction to times $t' < t_0$,

$$\begin{tikzcd} x(t) \arrow[r, "\beta"] \arrow[d, "\alpha^k"'] & \bar{x}(t-\theta_\beta) \\ x(t-k\theta_\alpha) \arrow[r, "\beta_-"] & \bar{x}(t-k\theta_\alpha-\theta_\beta) \arrow[u, "(\alpha^k)^*"'] \end{tikzcd} \tag{6.5}$$

Starting from $x(t)$, we apply α k times, so that $t - k\theta_\alpha < t_0$. This permits the application of β_-, i.e., β restricted to times preceding t_0. Applying the contrapositive of α k times, we arrive at $x(t) \to \bar{x}(t-\theta_\beta)$, which is β at time $t > t_0$. Thus β is preserved by α and β_-, becoming redundant after $t > t_0$. ∎

Remark. Any such theorem can only apply vacuously to conservative BDEs, which are reversible and do not have transients.

Theorem 6.1 can be applied immediately to

$$x(t) = x(t-\theta_1)\,\bar{x}(t-\theta_2) \tag{6.6}$$

where either $0 \leqslant \theta_1 \leqslant \theta_2 = 1$ or $0 \leqslant \theta_2 \leqslant \theta_1 = 1$. In this case,

$$\alpha: x(t) \to x(t-\theta_1)$$

$$\beta: x(t) \to \bar{x}(t-\theta_2)$$

and solutions of (6.6) must be determined asymptotically by

$$x(t) = x(t-\theta_1) \tag{6.7}$$

Further details on solutions of (6.6), (6.7), especially as θ_2 "passes through" θ_1, can be found in AM, Section 6.2.

Corollary 6.1. Let f in (6.1) carry implications $\alpha: x(t) \to x(t-\theta_\alpha)$ and $\beta: x(t) \to \bar{P}$, where P is a function of delayed values of x,

$u_i(t)=x(t-\theta_i)$, and assume that if $(a_1,\ldots,a_k)$ and $(b_1,\ldots,b_k)$ are sets of arguments for P with $a_i \to b_i$, $1 \leqslant i \leqslant k$, then $P(a_1,\ldots,a_k) \to P(b_1,\ldots,b_k)$. It follows that the solutions of (6.1) solve eventually an equation which does not contain β.

Proof. The property of P in the corollary characterizes the largest class of Boolean functions for which a diagram similar to (6.5), namely,

$$\begin{tikzcd} x(t) \arrow[r, "\beta"] \arrow[d, "\alpha^k"'] & \bar{P}(\mathbf{u}(t)) \\ x(t-k\theta_\alpha) \arrow[r, "\beta_-"] & \bar{P}(\mathbf{u}(t-k\theta_\alpha)) \arrow[u, "(\alpha^*)^k"'] \end{tikzcd}$$

commutes in the appropriate way. ■

Remark. The class of P to which the corollary applies contains all simple products and simple sums, e.g., $P(a_1,\ldots,a_k)=a_1a_2\ldots a_k$ or $P=a_1+a_2+\cdots+a_k$, and in fact represents a large class of connectives.

We see that interesting and rather complete information about asymptotic reduction of dissipative system behavior can be obtained from the formal representation of connectives by a grammar G of implications. We shall apply now this point of view to the other general situation considered, that of periodic solutions.

6.3. Periodicity and Structural Stability

In Theorem 6.1 and Corollary 6.1 we saw that all solutions of a BDE can be eventually periodic, and that the common periodic segment of the solutions satisfies a "reduced" or simplified equation. Next, a situation will be illustrated in which the common period can be determined, but no reduction exists.

Theorem 6.2. The equation

$$\bar{x}(t)=x(t-\theta_1)\,x(t-\theta_2) \tag{6.8}$$

has eventually periodic solutions of period $\pi=\theta_1+\theta_2$ for all values of the delays θ_1 and θ_2.

Remark. The notation of (6.8) is obvious shorthand, using De Morgan's rule [compare Eqs. (6.2), (6.3)].

Proof. Any equality $p=q$ is equivalent to two implications, $p \to q$ and $q \to p$, or $p \to q$ and $\bar{p} \to \bar{q}$. Thus (6.8) is equivalent to

$$\bar{x}(t) \to x(t-\theta_1)\,x(t-\theta_2) \tag{6.9a}$$

and

$$x(t) \to \bar{x}(t-\theta_1) + \bar{x}(t-\theta_2) \tag{6.9b}$$

Applying (6.9a) to each term on the right-hand side of (6.9b), and putting the right-hand side of the result into its CNF, yields

$$x(t) \to x(t-\theta_1-\theta_2) \tag{6.10}$$

A similar manipulation of the DNF completes the proof. ■

Remarks. (1) Implicit in the proof is also an upper bound on the *length of the transient*, $\lambda(\theta_1, \theta_2)$, namely, $\lambda \leqslant \theta_1 + \theta_2$. Indeed, this is greater than or equal to any of the delays needed for the arguments above. With arbitrary initial data on $0 \leqslant t \leqslant 1$, solutions will be periodic for $t \geqslant 1 + \lambda$.

(2) Notice that in this case *all* solutions are periodic, independently of whether θ_1 and θ_2 are rational or not.

Corollary 6.2. All solutions of

$$\bar{x}(t) = \prod_1^n x(t-\theta_k) \tag{6.11}$$

are eventually periodic with period

$$\pi(\boldsymbol{\theta}) = \sum_1^n \theta_k \qquad \text{for } n \text{ even} \tag{6.12a}$$

$$= 2\sum_1^n \theta_k \qquad \text{for } n \text{ odd} \tag{6.12b}$$

The length of transients, $\lambda(\boldsymbol{\theta})$, is bounded by

$$\lambda(\boldsymbol{\theta}) \leqslant \pi(\boldsymbol{\theta}) \tag{6.13}$$

Proof. See AM. ■

Equations (6.8), (6.11) give an example of BDEs for which, according to Theorem 5.2, the period function $\pi(\boldsymbol{\theta})$ is bounded away from 0 and from ∞, and hence continuous in the delays $\boldsymbol{\theta}$. As we shall see, these equations are also structurally stable.

More generally, in proceeding to a discussion of *structural stability*, we are interested in small deformations of BDEs leading to small deformations in their solutions. A BDE can be changed in two ways: by changing the connective f, and by changing the delays $\boldsymbol{\theta}$. Changes in f have to be measured in a discrete topology, and thus cannot be "small." The only small changes possible, measurable in any equivalent metric on $\mathbb{R}^n$, are those in $\boldsymbol{\theta}$. Hence it suffices for structural stability to consider small perturbations of the delays.

The concept of structural stability for BDEs will be patterned after that for differentiable dynamical systems (DDS[26,30,31]). Two systems on a topological space X are said to be topologically orbitally equivalent if there exists a homeomorphism $h: X \to X$ mapping orbit curves from one system to those of the other. A system is structurally stable if there is a neighborhood about it in the space of all systems, $\mathscr{S}(X)$, such that all systems in the neighborhood are topologically orbitally equivalent.

A necessary condition for structural stability of a BDE, like for a DDS, is that it have only properties which hold for a dense set of like systems. In the case of BDEs, such a property is that of having only eventually periodic solutions. Further reflection, inspired by Theorem 6.2 and Corollary 6.2, also shows that a dense set of BDEs has compact forward orbits, i.e., finite-length transients leading into periodic orbits.

These two topological properties reveal that a structurally stable system must have finite-length transients and eventually periodic orbits. In fact, there must be a bound on both the transient length and the period length over some neighborhood of the system. The remainder of this section is dedicated to a converse of this observation.

Suppose $\lambda(\boldsymbol{\theta})$ and $\pi(\boldsymbol{\theta})$ are nonnegative continuous functions of the delays on some neighborhood $U(\boldsymbol{\theta}_0)$ of $\boldsymbol{\theta}_0$, for some equation (6.1). Here $\lambda(\boldsymbol{\theta})$ is the upper bound for transients and $\pi(\boldsymbol{\theta})$ is the longest period, i.e., the smallest number which is a period of all solutions of (6.1) with delays $\boldsymbol{\theta}$.

Assume furthermore w.l.o.g. that $\boldsymbol{\theta}_0$ is irrational, or that it satisfies a nonresonance condition of a certain order, so that $q = \text{l.c.d.}\{\boldsymbol{\theta}_0\}$ is large compared to $\lambda_0 + \pi_0 \equiv \lambda(\boldsymbol{\theta}_0) + \pi(\boldsymbol{\theta}_0)$. Then one can choose a smaller neighborhood of $\boldsymbol{\theta}_0$, $V(\boldsymbol{\theta}_0) \subset U(\boldsymbol{\theta}_0)$, such that the common denominators of rational delays in V are all large compared to $\lambda_0 + \pi_0$. This is clearly possible since there are only finitely many points in U satisfying resonances of an order lower than $\boldsymbol{\theta}_0$.

It follows that the ordering in time of the delay lattice Γ of (6.1) for times $t \leqslant T = \sup_{\boldsymbol{\theta} \in V}\{\lambda(\boldsymbol{\theta}) + \pi(\boldsymbol{\theta})\}$ is independent of $\boldsymbol{\theta}$ for $\boldsymbol{\theta} \in V$, i.e., given any two points $(m_k \theta_k)$ and $(n_k \theta_k)$ in Γ, the inequality $\sum_k m_k \theta_k < \sum_k n_k \theta_k$, say, is preserved for any $\boldsymbol{\theta} = (\theta_k)$ in V, provided $\sum_k n_k \theta_k \leqslant T$. Hence an assignment of values (or jumps) to Γ for $\boldsymbol{\theta}_0$ is preserved in lattice coordinates for any other $\boldsymbol{\theta} \in V$. The change in $\boldsymbol{\theta}$ is reflected in the actual solution by a slight displacement of the jumps, but no collision between jumps, and subsequent large change in the solution, can occur.

We have thus constructed a homeomorphism from the BDE (6.1) with delays $\boldsymbol{\theta}_0$ to that with delays $\boldsymbol{\theta} \in V$: the map on Γ is the identity, and hence continuous and invertible in the discrete topology on Γ, while the map from Γ to the solutions is continuous by Theorem 2.2 and obviously invertible. Hence the composition of these two maps is continuous and invertible. This gives our structural stability theorem.

Theorem 6.3. Necessary and sufficient conditions for a BDE (6.1) with delays $\boldsymbol{\theta}_0$ to be structurally stable are that, for some neighborhood U of $\boldsymbol{\theta}_0$ in R^n: (1) all transients be bounded over U, (2) all periods be bounded over U,

$$\sup_{\boldsymbol{\theta} \in U} \{\lambda(\theta) + \pi(\theta)\} \leqslant M < \infty \tag{6.14}$$

Remarks. (1) This result holds for arbitrary systems of BDEs as well. All ideas for the proof are the same. [Compare also Eqs. (7) and (25a), in BDE I, with $\lambda(\boldsymbol{\theta}) = 0$.]

(2) It would be nice to have a purely algebraic characterization of structurally stable systems, in the same way that partial linearity provided a large class of BDEs with aperiodic solutions.

It is clear from the proof of Theorem 6.3 that classes of topological orbital equivalence for structurally stable BDEs are separated by resonances among delays, and that such resonances require sufficiently long periods or transients. Considering constant solutions as being periodic of period zero, it also follows that Hopf-type bifurcation in BDEs has to be related to delays "passing through each other."

6.4. Asymptotic Stability and Quasiperiodicity

In this section, we shall address two remaining types of behavior for BDEs: asymptotically stable constant solutions, and quasiperiodic solutions. Furthermore, the asymptotic behavior of Eq. (3.9) (Fig. 3) will also be discussed.

Consider first the equation

$$x(t) = x(t - \theta)\, x(t - 1) \tag{6.15}$$

with $0 < \theta < 1$ irrational. Aside from the constant solution $x^{(1)}(t) \equiv 1$, all solutions are eventually equal to the other constant solution $x^{(0)}(t) \equiv 0$. The *asymptotic stability* of $x^{(0)}(t) \equiv 0$ and instability of $x^{(1)}(t) \equiv 1$ follow by a study of the Liapunov function

$$L(t; x(t; \theta)) = \int_t^{t+1} x(s)\, ds \tag{6.16a}$$

Differentiating

$$L(t) = \int_t^{t+1} x(s - \theta)\, x(s - 1)\, ds$$

immediately gives

$$\frac{dL}{dt} = x(t)\,x(t-\theta+1) - x(t)$$

$$= x(t)[x(t-\theta+1)-1] \tag{6.16b}$$

and hence $dL/dt \leqslant 0$. Equality holds only when $x(t)=1$ almost everywhere.

Theorem 6.4. Given rationally unrelated delays $\boldsymbol{\theta} = (\theta_k)$, the BDE

$$x(t) = \prod_1^n x(t-\theta_k) \tag{6.17a}$$

has $x(t) \equiv 0$ as an asymptotically stable solution, while for the BDE

$$y(t) = \sum_1^n y(t-\theta_k) \tag{6.17b}$$

$y(t) \equiv 1$ is asymptotically stable.

Proof. See AM.

It is true that for smaller and smaller $\varepsilon = \int_0^1 (1-x(t))\,dt$, the transient length in (6.15) can be larger and larger. Furthermore, for a given ε sufficiently small, one can find a sequence of irrational delays θ_n such that the transient length increases without bound.

Theorem 6.3 implies therefore that Eq. (6.15) is not structurally stable. In fact, $\pi(\theta) = 0$ for all irrational θ, while for each rational $\theta = p/q$ periodic solutions of period q^{-1} exist. Thus $\pi(\theta)$ is not bounded away from zero, and Theorem 5.1 does not apply either.

We turn next to *quasiperiodic* behavior. A system of BDEs with quasiperiodic solutions is

$$\begin{aligned} x_1(t) &= x_1(t-\theta_{11})\,\bar{x}_3(t-\theta_{13}) \\ x_2(t) &= x_2(t-\theta_{22})\,\bar{x}_3(t-\theta_{23}) \\ x_3(t) &= x_1(t-\theta_{31})\,x_2(t-\theta_{32})\,x_3(t-\theta_{33}) \end{aligned} \tag{6.18}$$

where the delays $\{\theta_{ij}\}$ are taken to be rationally independent.

For initial data with $x_3(t) \equiv 0$, several types of solution are possible: (1) Periodic solutions of periods θ_{11} or θ_{22} obtain when $x_2(t)$ or $x_1(t)$ is constant in the initial data, respectively [see Eq. (6.7)]; (2) quasiperiodic solutions obtain when x_1 and x_2 have both nonconstant initial data. The presence of any resonance between θ_{11} and θ_{22} of course produces a periodic solution with period equal to l.c.m.$\{\theta_{11}, \theta_{22}\}$.

In view of the freedom involved in the choice of the resonance, periodic solutions of (6.18) with arbitrarily long period exist. According to Theorem 6.3, quasiperiodic solutions are therefore not structurally stable, in agreement with the situation for smooth dynamical systems (DDS).

For $\|x_3\| = \int_0^1 x_3(t)\,dt \neq 0$, the question of solution behavior becomes more involved. We start by noticing that $x_3(\tau) = 0$ implies $x_3(\tau + n\theta_{33}) = 0$, and likewise $x_j(\tau) = 0$ implies $x_3(\tau + n\theta_{3j}) = 0$ for $j = 1, 2$. Thus $x_3(t)$ "tends to zero," which tends to reduce system solutions to the previous situation. But for any $x_3(\tau) \neq 0$, the behavior of $x_j(\tau + \theta_{j3})$, $j = 1, 2$, will be disturbed, unless $x_j(\tau + \theta_{j3} - \theta_{jj}) = 0$ already. Clearly, given suitable inequalities among the delays, one can choose initial data for x_1, x_2, and x_3 such that $\|x_3\| \neq 0$ and the relations above are satisfied, recovering quasiperiodic solutions, i.e., $x_3(t)$ becomes eventually zero without affecting the behavior of $x_1(t)$ and $x_2(t)$.

Any point $0 \leqslant \tau < 1$ at which $x_3(\tau) = 1$ and the compatibility relations above are not satisfied generates a point at which x_1 or x_2 become zero, and these points are repeated with period θ_{11} or θ_{22} respectively. The rates at which the support of x_1 and x_2 in their respective periods is reduced depend on the rates at which various "slices" of the delay lattice Γ generate times dense on the line. In particular, $\int_t^{t+\theta_{jj}} x_j(s)\,ds$ could go to zero faster for $j = 1$ or $j = 2$ than for $j = 3$.

Notice in fact that the Liapunov function of (6.18),

$$L(t; \mathbf{x}(t; \boldsymbol{\theta})) = \int_t^{t+1} \sum_1^3 x_j(s)\,ds$$

yields $dL/dt \leqslant 0$, so that the trivial solution $x_1 \equiv x_2 \equiv x_3 \equiv 0$ is orbitally stable for any $\boldsymbol{\theta}$. Asymptotic stability, however, would depend on the delays, as indicated by an obvious generalization of Theorem 6.4.

We turn now to the case of Eq. (3.9). The normal forms of the right-hand side are $(p+q)(\bar{p}+\bar{q})\bar{r} = p\bar{q}\bar{r} + \bar{p}q\bar{r}$, with $p = x(t-1)$, $q = x(t-\theta)$ and $r = x(t-\tau)$. Both the numerical evidence of Fig. 3 and the discussion of Eq. (6.6) according to Theorem 6.1 suggest that the asymptotic behavior of solutions to (3.9) should be governed by its linear part, i.e., Eq. (3.5).

In fact, if the delays θ and τ are rational, the eventual periodicity yields the implications necessary to apply Theorem 6.1. Hence for all rational delays, the periods are those given by Eq. (3.5). These periods, however, increase with $q = \text{l.c.d.}\{\theta, \tau\}$, according to Theorem 5.4 of AM.

It follows, by Theorem 6.3, that Eq. (3.9) cannot be structurally stable, and we cannot conclude from asymptotic linearity for all rational delays that it is asymptotically linear, independently of delay values. Still, we suspect by another type of argument for asymptotic simplification, currently under study, that (3.5) eventually governs solutions of (3.9), independently of delays.

More generally, we see that asymptotic linearity, like asymptotic stability, tends to be metrically pervasive in BDEs, and structurally unstable. It is possible that solutions of perpetually decreasing complexity, like those of perpetually increasing complexity of Section 3, occur and can play an interesting role in the theory, as well as in applications.

7. CONCLUDING REMARKS

We have studied a class of dynamical systems with discrete variables and continuous time dependence. This mathematical study was motivated by the desire to create an appropriate framework for the qualitative investigation of complex biological and physical phenomena exhibiting threshold behavior, as well as distinct interaction times among the dependent quantities. In the present section, we shall recapitulate our main results, and outline some possible extensions and applications.

The most striking fact about BDEs is the existence of *aperiodic solutions with increasing complexity*, discovered numerically in BDE I, and proven here in Section 3. Equally surprising is the fact that these occur for *linear* and partially linear BDEs. Thus linearity over $\mathbb{Z}_2$ has interesting consequences. An intuitive way of explaining this phenomenon is the competition of conservativity, or its equivalents—reversibility in time or absence of transients, with saturation.

The next striking fact is that aperiodic solutions, which occur for rationally unrelated delays, can be approximated for increasingly long times by *periodic solutions* which obtain when all delays are rational (Section 4). Thus periodic and aperiodic solutions cannot be distinguished on the basis of experiments or numerical computations performed in finite time. In Section 5, and in Section 5.3 of AM, it was shown that period length increases exponentially with $q = 1/\Delta t$, where Δt is the maximum resolution in time.

Finally, in Section 6 we showed that asymptotic simplification, as well as asymptotic complexification, of solution behavior in BDEs is possible. *Structural stability* of a BDE was defined, and it was shown that it is equivalent to a bound on both period length and transient length for nearby BDEs. It follows that quasiperiodic solutions, as well as aperiodic solutions, which obtain for irrational delays, are metrically pervavise, but topologically vanishing for BDEs. The relation between a bifurcation theory for these equations, and the *resonances* between delays analyzed in

Section 4 in connection with solution *intermittency*, was also discussed. The study of intermittency and resonances in BDEs offers interesting connections with number theory (see Fig. 8 and Appendix B of AM).

Where do we go from here? The reduction results of Section 6 suggest that, given an arbitrary connective on many Boolean variables, it should be possible in general to obtain a much simpler connective, on considerably fewer variables, which governs the behavior of solutions for large times. This corresponds to an explicit, simple description of the global *limit set* for solutions, an elusive objective for (ODEs or PDEs),[32–34] but apparently feasible for BDEs.

Next, one can and should envisage the analysis of infinitely many Boolean variables, by analogy with PDEs and infinite cellular automata. The classification of what we could call now *ordinary* BDEs into conservative and dissipative (Section 2) suggests that *partial* BDEs of different types exist.

We notice here in passing that the discussion in the main text and in this section had been restricted to autonomous systems. This was done merely for the sake of brevity and convenience. Forcing, constant or variable in time and space, can be easily introduced and should play an important role in applications.

Among these, we shall only mention the applications which were the immediate motivation of BDE I, taken from theoretical climate dynamics.[35] The complexity of the *climatic system* on various time and space scales has led investigators to consider climate models incorporating two or three components of the system, and some of their possible interactions, at one time. Among the components are the atmosphere, biosphere, cryosphere, hydrosphere, lithosphere, and mantle. Their possible interactions include the ice-albedo feedback, CO_2 chemistry, the precipitation–temperature feedback, bedrock response to ice load, sea-ice effects on deep-water formation and many others (Ghil and Childress,[36] Part IV; Saltzmann[37]).

Classical approaches to modeling all these interactions, by systems of ODEs or PDEs, have great difficulty to master the climatic system's complexity with any thoroughness. At the same time, so-called *conceptual models* are used by paleoclimatologists to explain some of their observational findings. These models, while trying to be more comprehensive than the classical ones, cannot apply rigorous criteria in what is basically an *a posteriori* justification of an *a priori* supposition suggested by the data.

In this context, and in many other disciplines, such as various branches of the biosciences[38] and social sciences, precise mathematical reasoning is only starting to be applied to complex phenomena which are still incompletely measured and understood. For such disciplines, BDEs could become a systematic way of exploring formal conceptual models of the phenomena of interest.

Specific examples are currently being worked out in climate dynamics,[39–41] as well as in clinical biology.[42] It is relatively easy, in these examples and subsequent ones we plan to study, to determine from direct observations of the natural system, or from proxy data on it, the relative magnitude of the most important characteristic interaction times. Classical, ODE or PDE models require, on the other hand, the exact or at least approximate values of many other parameters, which might or might not affect qualitative system behavior. Therefore, a preliminary investigation using BDEs can be very useful in preparing the ground for and facilitating the task of more detailed analyses using ODEs, PDEs, and large numerical models.

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